

LECTURES ON SPIN REPRESENTATION THEORY OF SYMMETRIC GROUPS

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ABSTRACT. The representation theory of the symmetric groups is intimately related to geometry, algebraic combinatorics, and Lie theory. The spin representation theory of the symmetric groups was originally developed by Schur. In these lecture notes, we present a coherent account of the spin counterparts of several classical constructions such as the Frobenius characteristic map, Schur duality, the coinvariant algebra, Kostka polynomials, and Young's seminormal form.

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1. INTRODUCTION

1.1. The representation theory of symmetric groups has many connections and applications in geometry, combinatorics and Lie theory. The following classical constructions in representation theory of symmetric groups over the complex field \mathbb{C} are well known:

- (1) The characteristic map and symmetric functions
- (2) Schur duality
- (3) The coinvariant algebra
- (4) Kostka numbers and Kostka polynomials
- (5) Seminormal form representations and Jucys-Murphy elements

(1) and (2) originated in the work of Frobenius and Schur, (3) was developed by Chevalley (see also Steinberg [S], Lusztig [Lu1], and Kirillov [Ki]). The Kostka polynomials in (4) have striking combinatorial, geometric and representation theoretic interpretations, due to Lascoux, Schützenberger, Lusztig, Brylinski, Garsia and Procesi [LS, Lu2, Br, GP]. Young's seminormal form construction of irreducible modules of symmetric groups has been redone by Okounkov and Vershik [OV] using Jucys-Murphy elements.

Motivated by projective (i.e., spin) representation theory of finite groups and in particular of symmetric groups \mathfrak{S}_n , Schur [Sch] introduced a double cover $\tilde{\mathfrak{S}}_n$ of \mathfrak{S}_n :

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \tilde{\mathfrak{S}}_n \longrightarrow \mathfrak{S}_n \longrightarrow 1.$$

Let us write $\mathbb{Z}_2 = \{1, z\}$. The spin representation theory of \mathfrak{S}_n , or equivalently, the representation theory of the spin group algebra $\mathbb{C}\mathfrak{S}_n^- = \mathbb{C}\tilde{\mathfrak{S}}_n / \langle z + 1 \rangle$, has been systematically developed by Schur (see Józefiak [Jo1] for an excellent modern exposition via a superalgebra approach; also see Stembridge [St]).

The goal of these lecture notes is to provide a systematic account of the spin counterparts of the classical constructions (1)-(5) above over \mathbb{C} . Somewhat surprisingly, several of these spin analogues have been developed only very recently (see for example [WW2]). It is our hope that these notes will be accessible to people working in algebraic combinatorics who are interested in representation theory and to people in super representation theory who are interested in applications.

In addition to the topics (1)-(5), there are spin counterparts of several classical basic topics which are not covered in these lecture notes for lack of time and space: the Robinson-Schensted-Knuth correspondence (due to Sagan and Worley [Sag, Wor]; also see [GJK] for connections to crystal basis); the plactic monoid (Serrano [Ser]); Young symmetrizers [Naz, Se2]; Hecke algebras [Ol, JN, Wa1, Wa2]. We refer an interested reader to these papers and the references therein for details.

Let us explain the contents of the lecture notes section by section.

1.2. In Section 2, we explain how Schur's original motivation of studying the projective representations of the symmetric groups leads one to study the representations of the spin symmetric group algebras. It has become increasingly well known (cf. [Jo2, Se2, St, Ya] and [Kle, Chap. 13]) that the representation theory of spin symmetric group (super)algebra $\mathbb{C}\mathfrak{S}_n^-$ is super-equivalent to its counterpart for Hecke-Clifford (super)algebra $\mathcal{H}_n = \mathcal{C}l_n \rtimes \mathbb{C}\mathfrak{S}_n$. We shall explain such a super-equivalence in detail, and then we mainly work with the algebra \mathcal{H}_n , keeping in mind that the results can be transferred to the setting for $\mathbb{C}\mathfrak{S}_n^-$. We review the basics on superalgebras as needed.

The Hecke-Clifford superalgebra \mathcal{H}_n is identified as a quotient of the group algebra of a double cover \tilde{B}_n of the hyperoctahedral group B_n , and this allows us to apply various standard finite group constructions to the study of representation theory of \mathcal{H}_n . In particular, the split conjugacy classes for \tilde{B}_n (due to Read [Re]) are classified.

1.3. It is well known that the Frobenius characteristic map serves as a bridge to relate the representation theory of symmetric groups to the theory of symmetric functions.

In Section 3, the direct sum R^- of the Grothendieck groups of $\mathcal{H}_n\text{-mod}$ for all n is shown to carry a graded algebra structure and a bilinear form. Following Józefiak [Jo2], we formulate a spin version of the Frobenius characteristic map

$$\text{ch}^- : R^- \longrightarrow \Gamma_{\mathbb{Q}}$$

and establish its main properties, where $\Gamma_{\mathbb{Q}}$ is the ring of symmetric functions generated by the odd power-sums. It turns out that the Schur Q -functions Q_{ξ} associated to strict partitions ξ play the role of Schur functions, and up to some 2-powers, they correspond to the irreducible \mathcal{H}_n -modules D^{ξ} .

1.4. The classical Schur duality relates the representation theory of the general linear Lie algebras and that of the symmetric groups.

In Section 4, we explain in detail the Schur-Sergeev duality as formulated concisely in [Se1]. A double centralizer theorem for the actions of $\mathfrak{q}(n)$ and the Hecke-Clifford algebra \mathcal{H}_d on the tensor superspace $(\mathbb{C}^{n|n})^{\otimes d}$ is established, and this leads to an explicit multiplicity-free decomposition of the tensor superspace as a $U(\mathfrak{q}(n)) \otimes \mathcal{H}_d$ -module. As a consequence, a character formula for the simple $\mathfrak{q}(n)$ -modules appearing in the tensor superspace is derived in terms of Schur Q -functions. A more detailed exposition on materials covered in Sections 3 and 4 can be found in [CW, Chapter 3].

1.5. The symmetric group \mathfrak{S}_n acts on $V = \mathbb{C}^n$ and then on the symmetric algebra S^*V naturally. A closed formula for the graded multiplicity of a Specht module S^λ for a partition λ of n in the graded algebra S^*V in different forms has been well known (see Steinberg [S], Lusztig [Lu1] and Kirillov [Ki]). More generally, Kirillov and Pak [KP] obtained the bi-graded multiplicity of the Specht module S^λ for any λ in $S^*V \otimes \wedge^*V$ (see Theorem 5.4), where \wedge^*V denotes the exterior algebra. We give a new proof here by relating this bi-graded multiplicity to a 2-parameter specialization of the super Schur functions.

In Section 5, we formulate a spin analogue of the above graded multiplicity formulas. We present formulas with new proofs for the (bi)-graded multiplicity of a simple \mathcal{H}_n -module D^ξ in $\mathcal{C}l_n \otimes S^*V$, $\mathcal{C}l_n \otimes S^*V \otimes \wedge^*V$ and $\mathcal{C}l_n \otimes S^*V \otimes S^*V$ in terms of various specializations of the Schur Q -function $Q_\xi(z)$. The case of $\mathcal{C}l_n \otimes S^*V \otimes S^*V$ is new in this paper, while the other two cases were due to the authors [WW1]. The shifted hook formula for the principal specialization $Q_\xi(1, t, t^2, \dots)$ of $Q_\xi(z)$ was established by the authors [WW1] with a bijection proof and in a different form by Rosengren [Ro] based on formal Schur function identities. Here we present yet a third proof.

1.6. The Kostka numbers and Kostka(-Foulkes) polynomials are ubiquitous in combinatorics, geometry, and representation theory. Kostka polynomials have positive integer coefficients (see [LS] for a combinatorial proof, and see [GP] for a geometric proof). Kostka polynomials also coincide with Lusztig's q -weight multiplicity in finite-dimensional irreducible representations of the general linear Lie algebra [Lu2, Ka], and these are explained by a Brylinski-Kostant filtration on the weight spaces [Br]. More details can be found in the book of Macdonald [Mac] and the survey paper [DLT].

In Section 6, following a very recent work of the authors [WW2], we formulate a notion of spin Kostka polynomials, and establish their main properties including the integrality and positivity as well as interpretations in terms of representations of the Hecke-Clifford algebras and the queer Lie superalgebras. The graded multiplicities in the spin coinvariant algebra described in Section 5 are shown to be special cases of spin Kostka polynomials. Our constructions naturally give rise to formulations of the notions of spin Hall-Littlewood functions and spin Macdonald polynomials.

1.7. By studying the action of the Jucys-Murphy elements on the irreducible \mathfrak{S}_n -modules, Okounkov and Vershik [OV] developed a new approach to the representation theory of symmetric groups. In their approach, one can see the natural appearance of Young diagrams and standard tableaux, and obtain in the end Young's seminormal

form. A similar construction for the degenerate affine Hecke algebra associated to \mathfrak{S}_n has been obtained by Cherednik, Ram and Ruff [Ch, Ram, Ru].

In Section 7, we explain a recent approach to Young's seminormal form construction for the (affine) Hecke-Clifford algebra. The affine Hecke-Clifford algebra $\mathcal{H}_n^{\text{aff}}$ introduced by Nazarov [Naz] provides a natural general framework for \mathcal{H}_n . Following the independent works of Hill, Kujawa and Sussan [HKS] and the first author [Wan], we classify and construct the irreducible $\mathcal{H}_n^{\text{aff}}$ -modules on which the polynomial generators in $\mathcal{H}_n^{\text{aff}}$ act semisimply. A surjective homomorphism from $\mathcal{H}_n^{\text{aff}}$ to \mathcal{H}_n allows one to pass the results for $\mathcal{H}_n^{\text{aff}}$ to \mathcal{H}_n , and in this way we obtain Young's seminormal form for irreducible \mathcal{H}_n -modules. This recovers a construction of Nazarov [Naz] and the main result of Vershik-Sergeev [VS] who followed more closely Okounkov-Vershik's approach.

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2. SPIN SYMMETRIC GROUPS AND HECKE-CLIFFORD ALGEBRA

In this section, we formulate an equivalence between the spin representation theory of the symmetric group \mathfrak{S}_n and the representation theory of the Hecke-Clifford algebra \mathcal{H}_n . The algebra \mathcal{H}_n is then identified as a twisted group algebra for a distinguished double cover \tilde{B}_n of the hyperoctahedral group B_n . We classify the split conjugacy classes of \tilde{B}_n and show that the number of simple \mathcal{H}_n -modules is equal to the number of strict partitions of n .

2.1. From spin symmetric groups to \mathcal{H}_n . The symmetric group \mathfrak{S}_n is generated by the simple reflections $s_i = (i, i+1)$, $1 \leq i, j \leq n-1$, subject to the Coxeter relations:

$$(2.1) \quad s_i^2 = 1, \quad s_i s_j = s_j s_i, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad |i - j| > 1.$$

One of Schur's original motivations is the study of projective representations V of \mathfrak{S}_n , which are homomorphisms $\mathfrak{S}_n \rightarrow PGL(V) := GL(V)/\mathbb{C}^*$ (see [Sch]). By a sequence of analysis and deduction, Schur showed the study of projective representation theory (RT for short) of \mathfrak{S}_n is equivalent to the study of (linear) representation theory of a double cover $\tilde{\mathfrak{S}}_n$:

$$\text{Projective RT of } \mathfrak{S}_n \Leftrightarrow (\text{Linear}) \text{ RT of } \tilde{\mathfrak{S}}_n$$

A double cover $\tilde{\mathfrak{S}}_n$ means the following short exact sequence of groups (nonsplit for $n \geq 4$):

$$1 \longrightarrow \{1, z\} \longrightarrow \tilde{\mathfrak{S}}_n \xrightarrow{\pi_n} \mathfrak{S}_n \longrightarrow 1.$$

The quotient algebra $\mathbb{C}\mathfrak{S}_n^- = \widetilde{\mathbb{C}\mathfrak{S}_n}/\langle z+1 \rangle$ by the ideal generated by $(z+1)$ is called the *spin symmetric group algebra*. The algebra $\mathbb{C}\mathfrak{S}_n^-$ is an algebra generated by t_1, t_2, \dots, t_{n-1} subject to the relations:

$$t_i^2 = 1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad t_i t_j = -t_j t_i, \quad |i - j| > 1.$$

(A presentation for the group $\widetilde{\mathfrak{S}}_n$ can be obtained from the above formulas by keeping the first two relations and replacing the third one by $t_i t_j = z t_j t_i$.) $\mathbb{C}\mathfrak{S}_n^-$ is naturally a super (i.e., \mathbb{Z}_2 -graded) algebra with each t_i being odd, for $1 \leq i \leq n-1$.

By Schur's lemma, the central element z acts as ± 1 on a simple $\widetilde{\mathfrak{S}}_n$ -module. Hence we see that

$$\text{RT of } \widetilde{\mathfrak{S}}_n \Leftrightarrow \text{RT of } \mathfrak{S}_n \bigoplus \text{RT of } \mathbb{C}\mathfrak{S}_n^-$$

Schur then developed systematically the spin representation theory of \mathfrak{S}_n (i.e., the representation theory of $\mathbb{C}\mathfrak{S}_n^-$). We refer to Józefiak [Jo1] for an excellent modern exposition based on the superalgebra approach.

The development since late 1980's by several authors shows that the representation theory of $\mathbb{C}\mathfrak{S}_n^-$ is "super-equivalent" to the representation theory of a so-called Hecke-Clifford algebra \mathcal{H}_n :

$$(2.2) \quad \text{RT of } \mathbb{C}\mathfrak{S}_n^- \Leftrightarrow \text{RT of } \mathcal{H}_n$$

We will formulate this super-equivalence precisely in the next subsections.

2.2. A digression on superalgebras. By a vector superspace we mean a \mathbb{Z}_2 -graded space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. A superalgebra $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ satisfies $\mathcal{A}_i \cdot \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for $i, j \in \mathbb{Z}_2$. By an ideal I and a module M of a superalgebra \mathcal{A} in these lecture notes, we always mean that I and M are \mathbb{Z}_2 -graded, i.e., $I = (I \cap \mathcal{A}_{\bar{0}}) \oplus (I \cap \mathcal{A}_{\bar{1}})$, and $M = M_{\bar{0}} \oplus M_{\bar{1}}$ such that $\mathcal{A}_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}_2$. For a superalgebra \mathcal{A} , we let $\mathcal{A}\text{-mod}$ denote the category of \mathcal{A} -modules (with morphisms of degree one allowed). This superalgebra approach handles "self-associated and associates of simple modules" simultaneously in a conceptual way. There is a parity reversing functor Π on the category of vector superspaces (or module category of a superalgebra): for a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we let

$$\Pi(V) = \Pi(V)_{\bar{0}} \oplus \Pi(V)_{\bar{1}}, \quad \Pi(V)_i = V_{i+\bar{1}}, \forall i \in \mathbb{Z}_2.$$

Clearly, $\Pi^2 = \text{Id}$.

Given a vector superspace V with both even and odd subspaces of equal dimension and given an odd automorphism P of V of order 2, we define the following subalgebra of the endomorphism superalgebra $\text{End}(V)$:

$$Q(V) = \{x \in \text{End}(V) \mid x \text{ and } P \text{ super-commute}\}.$$

In case when $V = \mathbb{C}^{n|n}$ and P is the linear transformation in the block matrix form

$$\sqrt{-1} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

we write $Q(V)$ as $Q(n)$, which consists of $2n \times 2n$ matrices of the form:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where a and b are arbitrary $n \times n$ matrices, for $n \geq 0$. Note that we have a superalgebra isomorphism $Q(V) \cong Q(n)$ by properly choosing coordinates in V , whenever $\dim V = n|n$. A proof of the following theorem can be found in Józefiak [Jo] or [CW, Chapter 3].

Theorem 2.1 (Wall). *There are exactly two types of finite-dimensional simple associative superalgebras over \mathbb{C} : (1) the matrix superalgebra $M(m|n)$, which is naturally isomorphic to the endomorphism superalgebra of $\mathbb{C}^{m|n}$; (2) the superalgebra $Q(n)$.*

The basic results of finite-dimensional semisimple (unital associative) algebras over \mathbb{C} have natural super generalizations (cf. [Jo]). The proof is standard.

Theorem 2.2 (Super Wedderburn's Theorem). *A finite-dimensional semisimple superalgebra \mathcal{A} is isomorphic to a direct sum of simple superalgebras:*

$$\mathcal{A} \cong \bigoplus_{i=1}^m M(r_i|s_i) \oplus \bigoplus_{j=1}^q Q(n_j).$$

A simple \mathcal{A} -module V is annihilated by all but one such summand. We say V is of *type M* if this summand is of the form $M(r_i|s_i)$ and of *type Q* if this summand is of the form $Q(n_j)$. In particular, $\mathbb{C}^{r|s}$ is a simple module of the superalgebra $M(r|s)$ of type M, and $\mathbb{C}^{n|n}$ is a simple module of the superalgebra $Q(n)$. These two types of simple modules are distinguished by the following super analogue of Schur's Lemma (see [Jo], [CW, Chapter 3] for a proof).

Lemma 2.3. (*Super Schur's Lemma*) *If M and L are simple modules over a finite-dimensional superalgebra \mathcal{A} , then*

$$\dim \operatorname{Hom}_{\mathcal{A}}(M, L) = \begin{cases} 1 & \text{if } M \cong L \text{ is of type M,} \\ 2 & \text{if } M \cong L \text{ is of type Q,} \\ 0 & \text{if } M \not\cong L. \end{cases}$$

Remark 2.4. It can be shown (cf. [Jo]) that a simple module of type M as an ungraded module remains to be simple (which is sometimes referred to as “self-associated” in literature), and a simple module of type Q as an ungraded module is a direct sum of a pair of nonisomorphic simples (such pairs are referred to as “associates” in literature).

Given two associative superalgebras \mathcal{A} and \mathcal{B} , the tensor product $\mathcal{A} \otimes \mathcal{B}$ is naturally a superalgebra, with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|b| \cdot |a'|} (aa') \otimes (bb') \quad (a, a' \in \mathcal{A}, b, b' \in \mathcal{B}).$$

If V is an irreducible \mathcal{A} -module and W is an irreducible \mathcal{B} -module, $V \otimes W$ may not be irreducible (cf. [Jo], [BK], [Kle, Lemma 12.2.13]).

Lemma 2.5. *Let V be an irreducible \mathcal{A} -module and W be an irreducible \mathcal{B} -module.*

- (1) *If both V and W are of type M, then $V \otimes W$ is an irreducible $\mathcal{A} \otimes \mathcal{B}$ -module of type M.*
- (2) *If one of V or W is of type M and the other is of type Q, then $V \otimes W$ is an irreducible $\mathcal{A} \otimes \mathcal{B}$ -module of type Q.*
- (3) *If both V and W are of type Q, then $V \otimes W \cong X \oplus \Pi X$ for a type M irreducible $\mathcal{A} \otimes \mathcal{B}$ -module X .*

Moreover, all irreducible $\mathcal{A} \otimes \mathcal{B}$ -modules arise as components of $V \otimes W$ for some choice of irreducibles V, W .

If V is an irreducible \mathcal{A} -module and W is an irreducible \mathcal{B} -module, denote by $V \circledast W$ an irreducible component of $V \otimes W$. Thus,

$$V \otimes W = \begin{cases} V \circledast W \oplus \Pi(V \circledast W), & \text{if both } V \text{ and } W \text{ are of type } \mathbb{Q}, \\ V \circledast W, & \text{otherwise.} \end{cases}$$

Example 2.6. The Clifford algebra \mathcal{Cl}_n is the \mathbb{C} -algebra generated by $c_i (1 \leq i \leq n)$, subject to relations

$$(2.3) \quad c_i^2 = 1, \quad c_i c_j = -c_j c_i \quad (i \neq j).$$

Note that \mathcal{Cl}_n is a superalgebra with each generator c_i being odd, and $\dim \mathcal{Cl}_n = 2^n$.

For $n = 2k$ even, \mathcal{Cl}_n is isomorphic to a simple matrix superalgebra $M(2^{k-1}|2^{k-1})$. This can be seen by constructing an isomorphism $\mathcal{Cl}_2 \cong M(1|1)$ directly via Pauli matrices, and then using the superalgebra isomorphism

$$\mathcal{Cl}_{2k} = \underbrace{\mathcal{Cl}_2 \otimes \dots \otimes \mathcal{Cl}_2}_k.$$

Note that $\mathcal{Cl}_1 \cong Q(1)$. For $n = 2k + 1$ odd, we have superalgebra isomorphisms:

$$\mathcal{Cl}_n \cong \mathcal{Cl}_1 \otimes \mathcal{Cl}_{2k} \cong Q(1) \otimes M(2^{k-1}|2^{k-1}) \cong Q(2^k).$$

So \mathcal{Cl}_n is always a simple superalgebra, of type M for n even and of type Q for n odd. The fundamental fact that there are two types of complex Clifford algebras is a key to Bott's reciprocity.

2.3. A Morita super-equivalence. The symmetric group \mathfrak{S}_n acts as automorphisms on the Clifford algebra \mathcal{Cl}_n naturally by permuting the generators c_i . We will refer to the semi-direct product $\mathcal{H}_n := \mathcal{Cl}_n \rtimes \mathbb{C}\mathfrak{S}_n$ as the *Hecke-Clifford algebra*, where

$$(2.4) \quad s_i c_i = c_{i+1} s_i, \quad s_i c_{i+1} = c_i s_i, \quad s_i c_j = c_j s_i, \quad j \neq i, i+1.$$

Equivalently, $\sigma c_i = c_{\sigma(i)} \sigma$, for all $1 \leq i \leq n$ and $\sigma \in \mathfrak{S}_n$. The algebra \mathcal{H}_n is naturally a superalgebra by letting each $\sigma \in \mathfrak{S}_n$ be even and each c_i be odd.

Now let us make precise the super-equivalence (2.2).

By a direct computation, there is a superalgebra isomorphism (cf. [Se1, Ya]):

$$(2.5) \quad \begin{aligned} \mathbb{C}\mathfrak{S}_n^- \otimes \mathcal{Cl}_n &\longrightarrow \mathcal{H}_n \\ c_i &\mapsto c_i, \quad 1 \leq i \leq n, \\ t_j &\mapsto \frac{1}{\sqrt{-2}} s_j (c_j - c_{j+1}), \quad 1 \leq j \leq n-1. \end{aligned}$$

By Example 2.6, \mathcal{Cl}_n is a simple superalgebra. Hence, there is a unique (up to isomorphism) irreducible \mathcal{Cl}_n -module U_n , of type M for n even and of type Q for n odd. We have $\dim U_n = 2^k$ for $n = 2k$ or $n = 2k - 1$. Then the two exact functors

$$\begin{aligned} \mathfrak{F}_n &:= - \otimes U_n : \mathbb{C}\mathfrak{S}_n^- \text{-mod} \rightarrow \mathcal{H}_n \text{-mod}, \\ \mathfrak{G}_n &:= \text{Hom}_{\mathcal{Cl}_n}(U_n, -) : \mathcal{H}_n \text{-mod} \rightarrow \mathbb{C}\mathfrak{S}_n^- \text{-mod} \end{aligned}$$

define a Morita super-equivalence between the superalgebras \mathcal{H}_n and $\mathbb{C}\mathfrak{S}_n^-$ in the following sense.

Lemma 2.7. [BK, Lemma 9.9] [Kle, Proposition 13.2.2]

- (1) *Suppose that n is even. Then the two functors \mathfrak{F}_n and \mathfrak{G}_n are equivalences of categories with*

$$\mathfrak{F}_n \circ \mathfrak{G}_n \cong \text{id}, \quad \mathfrak{G}_n \circ \mathfrak{F}_n \cong \text{id}.$$

- (2) *Suppose that n is odd. Then*

$$\mathfrak{F}_n \circ \mathfrak{G}_n \cong \text{id} \oplus \Pi, \quad \mathfrak{G}_n \circ \mathfrak{F}_n \cong \text{id} \oplus \Pi.$$

Remark 2.8. The superalgebra isomorphism (2.5) and the Morita super-equivalence in Lemma 2.7 have a natural generalization to any finite Weyl group; see Khongsap-Wang [KW] (and the symmetric group case here is regarded as a type A case).

2.4. The group \tilde{B}_n and the algebra \mathcal{H}_n . Let Π_n be the finite group generated by a_i ($i = 1, \dots, n$) and the central element z subject to the relations

$$(2.6) \quad a_i^2 = 1, \quad z^2 = 1, \quad a_i a_j = z a_j a_i \quad (i \neq j).$$

The symmetric group \mathfrak{S}_n acts on Π_n by $\sigma(a_i) = a_{\sigma(i)}$, $\sigma \in \mathfrak{S}_n$. The semidirect product $\tilde{B}_n := \Pi_n \rtimes \mathfrak{S}_n$ admits a natural finite group structure and will be called the *twisted hyperoctahedral group*. Explicitly the multiplication in \tilde{B}_n is given by

$$(a, \sigma)(a', \sigma') = (a\sigma(a'), \sigma\sigma'), \quad a, a' \in \Pi_n, \sigma, \sigma' \in \mathfrak{S}_n.$$

Since $\Pi_n/\{1, z\} \simeq \mathbb{Z}_2^n$, the group \tilde{B}_n is a double cover of the hyperoctahedral group $B_n := \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$, and the order $|\tilde{B}_n|$ is $2^{n+1}n!$. That is, we have a short exact sequence of groups

$$(2.7) \quad 1 \longrightarrow \{1, z\} \longrightarrow \tilde{B}_n \xrightarrow{\theta_n} B_n \longrightarrow 1,$$

with $\theta_n(a_i) = b_i$, where b_i is the generator of the i th copy of \mathbb{Z}_2 in B_n . We define a \mathbb{Z}_2 -grading on the group \tilde{B}_n by setting the degree of each a_i to be 1 and the degree of elements in \mathfrak{S}_n to be 0. The group B_n inherits a \mathbb{Z}_2 -grading from \tilde{B}_n via the homomorphism θ_n .

The quotient algebra $\mathbb{C}\Pi_n/\langle z+1 \rangle$ is isomorphic to the Clifford algebra $\mathcal{C}l_n$ with the identification $\bar{a}_i = c_i$, $1 \leq i \leq n$. Hence we have a superalgebra isomorphism:

$$(2.8) \quad \mathbb{C}\tilde{B}_n/\langle z+1 \rangle \cong \mathcal{H}_n.$$

A \tilde{B}_n -module on which z acts as -1 is called a *spin \tilde{B}_n -module*. As a consequence of the isomorphism (2.8), we have the following equivalence:

$$\text{RT of } \mathcal{H}_n \Leftrightarrow \text{Spin RT of } \tilde{B}_n$$

2.5. The split conjugacy classes for B_n . Recall for a finite group G , the number of simple G -modules coincides with the number of conjugacy classes of G . The finite group B_n and its double cover \tilde{B}_n defined in (2.7) are naturally \mathbb{Z}_2 -graded. Since elements in a given conjugacy class of B_n share the same parity (\mathbb{Z}_2 -grading), it makes sense to talk about even and odd conjugacy classes of B_n (and \tilde{B}_n). One can show by using the Super Wedderburn's Theorem 2.2 that the number of simple \tilde{B}_n -modules coincides with the number of *even* conjugacy classes of \tilde{B}_n .

For a conjugacy class \mathcal{C} of B_n , $\theta_n^{-1}(\mathcal{C})$ is either a single conjugacy class of \tilde{B}_n or it splits into two conjugacy classes of \tilde{B}_n ; in the latter case, \mathcal{C} is called a *split* conjugacy class, and either conjugacy class in $\theta_n^{-1}(\mathcal{C})$ will also be called *split*. An element $x \in B_n$ is called *split* if the conjugacy class of x is split. If we denote $\theta_n^{-1}(x) = \{\tilde{x}, z\tilde{x}\}$, then x is split if and only if \tilde{x} is not conjugate to $z\tilde{x}$. By analyzing the structure of the even center of $\mathbb{C}\tilde{B}_n$ using the Super Wedderburn's Theorem 2.2 and noting that $\mathbb{C}\tilde{B}_n \cong \mathbb{C}B_n \oplus \mathcal{H}_n$, one can show the following [Jo] (also see [CW, Chapter 3]).

Proposition 2.9. (1) *The number of simple \mathcal{H}_n -modules equals the number of even split conjugacy classes of B_n .*
 (2) *The number of simple \mathcal{H}_n -modules of type Q equals the number of odd split conjugacy classes of B_n .*

Denote by \mathcal{P} the set of all partitions and by \mathcal{P}_n the set of partitions of n . We denote by \mathcal{SP}_n the set of all strict partitions of n , and by \mathcal{OP}_n the set of all odd partitions of n . Moreover, we denote

$$\mathcal{SP} = \bigcup_{n \geq 0} \mathcal{SP}_n, \quad \mathcal{OP} = \bigcup_{n \geq 0} \mathcal{OP}_n,$$

and denote

$$\begin{aligned} \mathcal{SP}_n^+ &= \{\lambda \in \mathcal{SP}_n \mid \ell(\lambda) \text{ is even}\}, \\ \mathcal{SP}_n^- &= \{\lambda \in \mathcal{SP}_n \mid \ell(\lambda) \text{ is odd}\}. \end{aligned}$$

The conjugacy classes of the group B_n (a special case of a wreath product) can be described as follows, cf. Macdonald [Mac, I, Appendix B]. Given a cycle $t = (i_1, \dots, i_m)$, we call the set $\{i_1, \dots, i_m\}$ the *support* of t , denoted by $\text{supp}(t)$. The subgroup \mathbb{Z}_2^n of B_n consists of elements $b_I := \prod_{i \in I} b_i$ for $I \subset \{1, \dots, n\}$. Each element $b_I \sigma \in B_n$ can be written as a product (unique up to reordering) $b_I \sigma = (b_{I_1} \sigma_1)(b_{I_2} \sigma_2) \dots (b_{I_k} \sigma_k)$, where $\sigma \in \mathfrak{S}_n$ is a product of disjoint cycles $\sigma = \sigma_1 \dots \sigma_k$, and $I_a \subset \text{supp}(\sigma_a)$ for each $1 \leq a \leq k$. The *cycle-product* of each $b_{I_a} \sigma_a$ is defined to be the element $\prod_{i \in I_a} b_i \in \mathbb{Z}_2$ (which can be conveniently thought as a sign \pm). Let m_i^+ (respectively, m_i^-) be the number of i -cycles of $b_I \sigma$ with associated cycle-product being the identity (respectively, the non-identity). Then $\rho^+ = (m_i^+)_{i \geq 1}$ and $\rho^- = (m_i^-)_{i \geq 1}$ are partitions such that $|\rho^+| + |\rho^-| = n$. The pair of partitions (ρ^+, ρ^-) will be called the *type* of the element $b_I \sigma$.

The basic fact on the conjugacy classes of B_n is that two elements of B_n are conjugate if and only if their types are the same.

Example 2.10. Let $\tau = (1, 2, 3, 4)(5, 6, 7)(8, 9), \sigma = (1, 3, 8, 6)(2, 7, 9)(4, 5) \in \mathfrak{S}_{10}$. Both $x = ((+, +, +, -, +, +, +, -, +, -), \tau)$ and $y = ((+, -, -, -, +, -, -, -, +, -), \sigma)$ in B_{10} have the same type $(\rho^+, \rho^-) = ((3), (4, 2, 1))$. Then x is conjugate to y in B_{10} .

The even and odd split conjugacy classes of B_n are classified by Read [Re] as follows. The proof relies on an elementary yet lengthy case-by-case analysis on conjugation, and it will be skipped (see [CW, Chapter 3] for detail).

Theorem 2.11. [Re] *The conjugacy class C_{ρ^+, ρ^-} in B_n splits if and only if*

- (1) *For even C_{ρ^+, ρ^-} , we have $\rho^+ \in \mathcal{OP}_n$ and $\rho^- = \emptyset$;*
- (2) *For odd C_{ρ^+, ρ^-} , we have $\rho^+ = \emptyset$ and $\rho^- \in \mathcal{SP}_n^-$.*

For $\alpha \in \mathcal{OP}_n$ we let \mathcal{C}_α^+ be the split conjugacy class in \tilde{B}_n which lies in $\theta_n^{-1}(C_{\alpha, \emptyset})$ and contains a permutation in \mathfrak{S}_n of cycle type α . Then $z\mathcal{C}_\alpha^+$ is the other conjugacy class in $\theta_n^{-1}(C_{\alpha, \emptyset})$, which will be denoted by \mathcal{C}_α^- . By (2.8) and Proposition 2.9, we can construct a (square) character table $(\varphi_\alpha)_{\varphi, \alpha}$ for \mathcal{H}_n whose rows are simple \mathcal{H}_n -characters φ or equivalently, simple spin \tilde{B}_n -characters (with \mathbb{Z}_2 -grading implicitly assumed), and whose columns are even split conjugacy classes \mathcal{C}_α^+ for $\alpha \in \mathcal{OP}_n$.

Recall the Euler identity that $|\mathcal{SP}_n| = |\mathcal{OP}_n|$. By Proposition 2.9 and Theorem 2.11, we have the following.

Corollary 2.12. *The number of simple \mathcal{H}_n -modules equals $|\mathcal{SP}_n|$. More precisely, the number of simple \mathcal{H}_n -modules of type M equals $|\mathcal{SP}_n^+|$ and the number of simple \mathcal{H}_n -modules of type Q equals $|\mathcal{SP}_n^-|$.*

3. THE (SPIN) CHARACTERISTIC MAP

In this section, we develop systematically the representation theory of \mathcal{H}_n after a quick review of the Frobenius characteristic map for \mathfrak{S}_n . Following [Jo2], we define a (spin) characteristic map using the character table for the simple \mathcal{H}_n -modules, and establish its main properties. We review the relevant aspects of symmetric functions. The image of the irreducible characters of \mathcal{H}_n under the characteristic map are shown to be Schur Q -functions up to some 2-powers.

3.1. The Frobenius characteristic map. The conjugacy classes of \mathfrak{S}_n are parameterized by partitions λ of n . Let

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$$

denote the order of the centralizer of an element in a conjugacy class of cycle type λ .

Let $R_n := R(\mathfrak{S}_n)$ be the Grothendieck group of $\mathfrak{S}_n\text{-mod}$, which can be identified with the \mathbb{Z} -span of irreducible characters χ^λ of the Specht modules S^λ for $\lambda \in \mathcal{P}_n$. There is a bilinear form on R_n so that $(\chi^\lambda, \chi^\mu) = \delta_{\lambda\mu}$. This induces a bilinear form on the direct sum

$$R = \bigoplus_{n=0}^{\infty} R_n,$$

so that the R_n 's are orthogonal for different n . Here $R_0 = \mathbb{Z}$. In addition, R is a graded ring with multiplication given by $fg = \text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(f \otimes g)$ for $f \in R_m$ and $g \in R_n$.

Denote by Λ the ring of symmetric functions in infinitely many variables, which is the \mathbb{Z} -span of the monomial symmetric functions m_λ for $\lambda \in \mathcal{P}$. There is a standard bilinear form (\cdot, \cdot) on Λ such that the Schur functions s_λ form an orthonormal basis for Λ . The ring Λ admits several distinguished bases: the complete homogeneous symmetric functions $\{h_\lambda\}$, the elementary symmetric functions $\{e_\lambda\}$, and the power-sum symmetric functions $\{p_\lambda\}$. See [Mac].

The (Frobenius) characteristic map $\text{ch} : R \rightarrow \Lambda$ is defined by

$$(3.1) \quad \text{ch}(\chi) = \sum_{\mu \in \mathcal{P}_n} z_\mu^{-1} \chi_\mu p_\mu,$$

where χ_μ denotes the character value of χ at a permutation of cycle type μ . Denote by $\mathbf{1}_n$ and sgn_n the trivial and the sign module/character of \mathfrak{S}_n , respectively. It is well known that

- ch is an isomorphism of graded rings.
- ch is an isometry.
- $\text{ch}(\mathbf{1}_n) = h_n$, $\text{ch}(\text{sgn}_n) = e_n$, $\text{ch}(\chi^\lambda) = s_\lambda$.

Moreover, the following holds for any composition μ of n :

$$(3.2) \quad \text{ch}(\text{ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_n} \mathbf{1}_\mu) = h_\mu,$$

where $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots$ denotes the associated Young subgroup.

We record the Cauchy identity for later use (cf. [Mac, I, §4])

$$(3.3) \quad \sum_{\mu \in \mathcal{P}} m_\mu(y) h_\mu(z) = \prod_{i,j} \frac{1}{1 - y_i z_j} = \sum_{\lambda \in \mathcal{P}} s_\lambda(y) s_\lambda(z).$$

3.2. The basic spin module. The exterior algebra $\mathcal{C}l_n$ is naturally an \mathcal{H}_n -module (called the *basic spin module*) where the action is given by

$$c_i \cdot (c_{i_1} c_{i_2} \cdots) = c_i c_{i_1} c_{i_2} \cdots, \quad \sigma \cdot (c_{i_1} c_{i_2} \cdots) = c_{\sigma(i_1)} c_{\sigma(i_2)} \cdots,$$

for $\sigma \in \mathfrak{S}_n$. Let $\sigma = \sigma_1 \cdots \sigma_\ell \in \mathfrak{S}_n$ be a cycle decomposition with cycle length of σ_i being μ_i . If I is a union of some of the $\text{supp}(\sigma_i)$'s, say $I = \text{supp}(\sigma_{i_1}) \cup \cdots \cup \text{supp}(\sigma_{i_s})$, then $\sigma(c_I) = (-1)^{\mu_{i_1} + \cdots + \mu_{i_s} - s} c_I$. Otherwise, $\sigma(c_I)$ is not a scalar multiple of c_I . This observation quickly leads to the following.

Lemma 3.1. *The value of the character ξ_α^n of the basic spin \mathcal{H}_n -module at the conjugacy class \mathcal{C}_α^+ is given by*

$$(3.4) \quad \xi_\alpha^n = 2^{\ell(\alpha)}, \quad \alpha \in \mathcal{OP}_n.$$

The basic spin module of \mathcal{H}_n should be regarded as the spin analogue of the trivial/sign modules of \mathfrak{S}_n .

3.3. The ring R^- . Thanks to the superalgebra isomorphism (2.8), $\mathcal{H}_n\text{-mod}$ is equivalent to the category of spin \tilde{B}_n -modules. We shall not distinguish these two isomorphic categories below, and the latter one has the advantage that one can apply the standard arguments from the theory of finite groups directly as we have seen in Section 2. Denote by R_n^- the Grothendieck group of $\mathcal{H}_n\text{-mod}$. As in the usual (ungraded) case, we may replace the isoclasses of modules by their characters, and then regard R_n^- as the free

abelian group with a basis consisting of the characters of the simple \mathcal{H}_n -modules. It follows by Corollary 2.12 that the rank of R_n^- is $|\mathcal{SP}_n|$. Let

$$R^- := \bigoplus_{n=0}^{\infty} R_n^-, \quad R_{\mathbb{Q}}^- := \bigoplus_{n=0}^{\infty} \mathbb{Q} \otimes_{\mathbb{Z}} R_n^-,$$

where it is understood that $R_0^- = \mathbb{Z}$.

We shall define a ring structure on R^- as follows. Let $\mathcal{H}_{m,n}$ be the subalgebra of \mathcal{H}_{m+n} generated by $\mathcal{C}l_{m+n}$ and $\mathfrak{S}_m \times \mathfrak{S}_n$. For $M \in \mathcal{H}_m\text{-mod}$ and $N \in \mathcal{H}_n\text{-mod}$, $M \otimes N$ is naturally an $\mathcal{H}_{m,n}$ -module, and we define the product

$$[M] \cdot [N] = [\mathcal{H}_{m,n} \otimes_{\mathcal{H}_{m,n}} (M \otimes N)],$$

and then extend by \mathbb{Z} -bilinearity. It follows from the properties of the induced characters that the multiplication on R^- is commutative and associative.

Given spin \tilde{B}_n -modules M, N , we define a bilinear form on R and so on $R_{\mathbb{Q}}$ by letting

$$(3.5) \quad \langle M, N \rangle = \dim \operatorname{Hom}_{\tilde{B}_n}(M, N).$$

3.4. The Schur Q -functions. The materials in this subsection are pretty standard (cf. [Mac, Jo1] and [CW, Appendix A]). Recall p_r is the r th power sum symmetric function, and for a partition $\mu = (\mu_1, \mu_2, \dots)$ we define $p_{\mu} = p_{\mu_1} p_{\mu_2} \dots$. Let $x = \{x_1, x_2, \dots\}$ be a set of indeterminates. Define a family of symmetric functions $q_r = q_r(x)$, $r \geq 0$, via a generating function

$$(3.6) \quad Q(t) := \sum_{r \geq 0} q_r(x) t^r = \prod_i \frac{1 + tx_i}{1 - tx_i}.$$

It follows from (3.6) that

$$Q(t) = \exp \left(2 \sum_{r \geq 1, r \text{ odd}} \frac{p_r t^r}{r} \right).$$

Componentwise, we have

$$(3.7) \quad q_n = \sum_{\alpha \in \mathcal{OP}_n} 2^{\ell(\alpha)} z_{\alpha}^{-1} p_{\alpha}.$$

Note that $q_0 = 1$, and that $Q(t)$ satisfies the relation

$$(3.8) \quad Q(t)Q(-t) = 1,$$

which is equivalent to the identities:

$$\sum_{r+s=n} (-1)^r q_r q_s = 0, \quad n \geq 1.$$

These identities are vacuous for n odd. When $n = 2m$ is even, we obtain that

$$(3.9) \quad q_{2m} = \sum_{r=1}^{m-1} (-1)^{r-1} q_r q_{2m-r} - \frac{1}{2} (-1)^m q_m^2.$$

Let Γ be the \mathbb{Z} -subring of Λ generated by the q_r 's:

$$\Gamma = \mathbb{Z}[q_1, q_2, q_3, \dots].$$

The ring Γ is graded by the degree of functions: $\Gamma = \bigoplus_{n \geq 0} \Gamma^n$. We set $\Gamma_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$. For any partition $\mu = (\mu_1, \mu_2, \dots)$, we define

$$q_{\mu} = q_{\mu_1} q_{\mu_2} \cdots$$

Theorem 3.2. *The following holds for Γ and $\Gamma_{\mathbb{Q}}$:*

- (1) $\Gamma_{\mathbb{Q}}$ is a polynomial algebra with polynomial generators p_{2r-1} for $r \geq 1$.
- (2) $\{p_{\mu} \mid \mu \in \mathcal{OP}\}$ forms a linear basis for $\Gamma_{\mathbb{Q}}$.
- (3) $\{q_{\mu} \mid \mu \in \mathcal{OP}\}$ forms a linear basis for $\Gamma_{\mathbb{Q}}$.
- (4) $\{q_{\mu} \mid \mu \in \mathcal{SP}\}$ forms a \mathbb{Z} -basis for Γ .

Proof. By clearing the denominator of the identity

$$\frac{Q'(t)}{Q(t)} = 2 \sum_{r \geq 0} p_{2r+1} t^{2r},$$

we deduce that

$$r q_r = 2(p_1 q_{r-1} + p_3 q_{r-3} + \dots).$$

By using induction on r , we conclude that (i) each q_r is expressible as a polynomial in terms of p_s 's with odd s ; (ii) each p_r with odd r is expressible as a polynomial in terms of q_s 's, which can be further restricted to the odd s . So, $\Gamma_{\mathbb{Q}} = \mathbb{Q}[p_1, p_3, \dots] = \mathbb{Q}[q_1, q_3, \dots]$, and from this (1), (2) and (3) follow.

To prove (4), it suffices to show that, for any partition λ ,

$$q_{\lambda} = \sum_{\mu \in \mathcal{SP}, \mu \geq \lambda} a_{\mu\lambda} q_{\mu},$$

for some $a_{\mu\lambda} \in \mathbb{Z}$. This can be seen by induction downward on the dominance order on λ with the help of (3.9). \square

We shall define the Schur Q -functions Q_{λ} , for $\lambda \in \mathcal{SP}$. Let

$$Q_{(n)} = q_n, \quad n \geq 0.$$

Consider the generating function

$$Q(t_1, t_2) := (Q(t_1)Q(t_2) - 1) \frac{t_1 - t_2}{t_1 + t_2}.$$

By (3.8), $Q(t_1, t_2)$ is a power series in t_1 and t_2 , and we write

$$Q(t_1, t_2) = \sum_{r, s \geq 0} Q_{(r, s)} t_1^r t_2^s.$$

Noting $Q(t_1, t_2) = -Q(t_2, t_1)$, we have $Q_{(r, s)} = -Q_{(s, r)}$, $Q_{(r, 0)} = q_r$. In addition,

$$Q_{(r, s)} = q_r q_s + 2 \sum_{i=1}^s (-1)^i q_{r+i} q_{s-i}, \quad r > s.$$

For a strict partition $\lambda = (\lambda_1, \dots, \lambda_m)$, we define the Schur Q -function Q_λ recursively as follows:

$$\begin{aligned} Q_\lambda &= \sum_{j=2}^m (-1)^j Q_{(\lambda_1, \lambda_j)} Q_{(\lambda_2, \dots, \hat{\lambda}_j, \dots, \lambda_m)}, \quad \text{for } m \text{ even,} \\ Q_\lambda &= \sum_{j=1}^m (-1)^{j-1} Q_{\lambda_j} Q_{(\lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_m)}, \quad \text{for } m \text{ odd.} \end{aligned}$$

Note that the Q_λ above is simply the Laplacian expansion of the pfaffian of the skew-symmetric matrix $(Q_{(\lambda_i, \lambda_j)})$ when m is even (possibly $\lambda_m = 0$).

It follows from the recursive definition of Q_λ and (3.9) that, for $\lambda \in \mathcal{SP}_n$,

$$Q_\lambda = q_\lambda + \sum_{\mu \in \mathcal{SP}_n, \mu > \lambda} d_{\lambda\mu} q_\mu,$$

for some $d_{\lambda\mu} \in \mathbb{Z}$. From this and Theorem 3.2 we further deduce the following.

Theorem 3.3. *The Q_λ for all strict partitions λ form a \mathbb{Z} -basis for Γ . Moreover, for any composition μ of n , we have*

$$q_\mu = \sum_{\lambda \in \mathcal{SP}_n, \lambda \geq \mu} \hat{K}_{\lambda\mu} Q_\lambda,$$

where $\hat{K}_{\lambda\mu} \in \mathbb{Z}$ and $\hat{K}_{\lambda\lambda} = 1$.

Let $x = \{x_1, x_2, \dots\}$ and $y = \{y_1, y_2, \dots\}$ be two independent sets of variables. We have by (3.6) that

$$(3.10) \quad \prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j} = \sum_{\alpha \in \mathcal{OP}} 2^{\ell(\alpha)} z_\alpha^{-1} p_\alpha(x) p_\alpha(y).$$

We define an inner product $\langle \cdot, \cdot \rangle$ on $\Gamma_{\mathbb{Q}}$ by letting

$$(3.11) \quad \langle p_\alpha, p_\beta \rangle = 2^{-\ell(\alpha)} z_\alpha \delta_{\alpha\beta}.$$

Theorem 3.4. *We have*

$$\langle Q_\lambda, Q_\mu \rangle = 2^{\ell(\lambda)} \delta_{\lambda\mu}, \quad \lambda, \mu \in \mathcal{SP}.$$

Moreover, the following Cauchy identity holds:

$$\prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j} = \sum_{\lambda \in \mathcal{SP}} 2^{-\ell(\lambda)} Q_\lambda(x) Q_\lambda(y).$$

We will skip the proof of Theorem 3.4 and make some comments only. The two statements therein can be seen to be equivalent in light of (3.10) and (3.11). One possible proof of the first statement following from the theory of Hall-Littlewood functions [Mac], and another direct proof is also available [Jo1]. The second statement would follow easily once the shifted Robinson-Schensted-Knuth correspondence is developed (cf. [Sag, Corollary 8.3]).

3.5. The characteristic map. We define the (spin) *characteristic map*

$$\text{ch}^- : R_{\mathbb{Q}}^- \longrightarrow \Gamma_{\mathbb{Q}}$$

to be the linear map given by

$$(3.12) \quad \text{ch}^-(\varphi) = \sum_{\alpha \in \mathcal{OP}_n} z_{\alpha}^{-1} \varphi_{\alpha} p_{\alpha}, \quad \varphi \in R_n^-.$$

The following theorem is due to Józeffiak [Jo2] (see [CW, Chapter 3] for an exposition).

Theorem 3.5. [Jo2] (1) *The characteristic map $\text{ch}^- : R_{\mathbb{Q}}^- \rightarrow \Gamma_{\mathbb{Q}}$ is an isometry.*

(2) *The characteristic map $\text{ch}^- : R_{\mathbb{Q}}^- \rightarrow \Gamma_{\mathbb{Q}}$ is an isomorphism of graded algebras.*

Sketch of a proof. We first show that ch^- is an isometry. Take $\varphi, \psi \in R_n^-$. Since φ is a character of a \mathbb{Z}_2 -graded module, we have the character value $\varphi_{\alpha} = 0$ for $\alpha \notin \mathcal{OP}_n$. We can reformulate the bilinear form (3.5) using the standard bilinear form formula on characters of the finite group \tilde{B}_n as

$$\langle \varphi, \psi \rangle = \sum_{\alpha \in \mathcal{OP}_n} 2^{-\ell(\alpha)} z_{\alpha}^{-1} \varphi_{\alpha} \psi_{\alpha},$$

which can be seen using (3.11) to be equal to $\langle \text{ch}^-(\varphi), \text{ch}^-(\psi) \rangle$.

Next, we show that ch^- is a homomorphism of graded algebras. For $\phi \in R_m^-$, $\psi \in R_n^-$ and $\gamma \in \mathcal{OP}_{m+n}$, we obtain a standard induced character formula for $(\phi \cdot \psi)_{\gamma}$ evaluated at a conjugacy class \mathcal{C}_{γ}^+ . This together with the definition of ch^- imply that

$$\begin{aligned} \text{ch}^-(\phi \cdot \psi) &= \sum_{\gamma \in \mathcal{OP}} z_{\gamma}^{-1} (\phi \cdot \psi)_{\gamma} p_{\gamma} \\ &= \sum_{\gamma} \sum_{\alpha, \beta \in \mathcal{OP}, \alpha \cup \beta = \gamma} z_{\gamma}^{-1} \frac{z_{\gamma}}{z_{\alpha} z_{\beta}} \phi_{\alpha} \psi_{\beta} p_{\gamma} = \text{ch}^-(\phi) \text{ch}^-(\psi). \end{aligned}$$

Recalling the definition of ch^- and the basic spin character ξ^n , it follows by (3.7) and Lemma 3.1 that $\text{ch}^-(\xi^n) = q_n$. Since q_n for $n \geq 1$ generate the algebra $\Gamma_{\mathbb{Q}}$ by Theorem 3.2, ch^- is surjective. Then ch^- is an isomorphism of graded vector spaces by the following comparison of the graded dimensions (cf. Corollary 2.12 and Theorem 3.2):

$$\dim_q R_{\mathbb{Q}}^- = \prod_{r \geq 1} (1 + q^r) = \dim_q \Gamma_{\mathbb{Q}}.$$

This completes the proof of the theorem. \square

Recall from the proof above that $\text{ch}^-(\xi^n) = q_n$. Regarding $\xi^{(n)} = \xi^n$, we define ξ^{λ} for $\lambda \in \mathcal{SP}$ using the same recurrence relations for the Schur Q -functions Q_{λ} . Then by Theorem 3.5, $\text{ch}^-(\xi^{\lambda}) = Q_{\lambda}$, and $\langle \xi^{\lambda}, \xi^{\mu} \rangle = 2^{\ell(\lambda)} \delta_{\lambda\mu}$, for $\lambda, \mu \in \mathcal{SP}$.

For a partition λ with length $\ell(\lambda)$, we set

$$(3.13) \quad \delta(\lambda) = \begin{cases} 0, & \text{if } \ell(\lambda) \text{ is even,} \\ 1, & \text{if } \ell(\lambda) \text{ is odd.} \end{cases}$$

By chasing the recurrence relation more closely, we can show by induction on $\ell(\lambda)$ that the element

$$\zeta^{\lambda} := 2^{-\frac{\ell(\lambda) - \delta(\lambda)}{2}} \xi^{\lambda}$$

lies in R^- , for $\lambda \in \mathcal{SP}_n$. Note that

$$(3.14) \quad \text{ch}^-(\zeta^\lambda) = 2^{-\frac{\ell(\lambda)-\delta(\lambda)}{2}} Q_\lambda.$$

It follows that, for each $\lambda \in \mathcal{SP}_n$,

$$Q_\lambda = 2^{\frac{\ell(\lambda)-\delta(\lambda)}{2}} \sum_{\alpha \in \mathcal{OP}_n} z_\alpha^{-1} \zeta_\alpha^\lambda p_\alpha.$$

Given $\mu \in \mathcal{P}_n$, let us denote $\mathcal{H}_\mu := \mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2} \otimes \cdots$, and recall the Young subgroup \mathfrak{S}_μ of \mathfrak{S}_n . The induced \mathcal{H}_n -module

$$M^\mu := \mathcal{H}_n \otimes_{\mathbb{C}\mathfrak{S}_\mu} \mathbf{1}_n$$

will be called a *permutation module* of \mathcal{H}_n . By the transitivity of the tensor product, it can be rewritten as

$$M^\mu = \mathcal{H}_n \otimes_{\mathcal{H}_\mu} (\mathcal{C}l_{\mu_1} \otimes \mathcal{C}l_{\mu_2} \otimes \cdots).$$

Since $\text{ch}^-(\xi^n) = q_n$ and ch^- is an algebra homomorphism, we obtain that

$$(3.15) \quad \text{ch}^-(M^\mu) = q_\mu.$$

Theorem 3.6. [Jo2] *The set of characters ζ^λ for $\lambda \in \mathcal{SP}_n$ is a complete list of pairwise non-isomorphic simple (super) characters of \mathcal{H}_n . Moreover, the degree of ζ^λ is equal to $2^{n-\frac{\ell(\lambda)-\delta(\lambda)}{2}} g_\lambda$, where*

$$g_\lambda = \frac{n!}{\lambda_1! \cdots \lambda_\ell!} \prod_{i < j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

Sketch of a proof. For strict partitions λ, μ , we have

$$(3.16) \quad \begin{aligned} \langle \zeta^\lambda, \zeta^\lambda \rangle &= \begin{cases} 1 & \text{for } \ell(\lambda) \text{ even,} \\ 2 & \text{for } \ell(\lambda) \text{ odd,} \end{cases} \\ \langle \zeta^\lambda, \zeta^\mu \rangle &= 0, \quad \text{for } \lambda \neq \mu. \end{aligned}$$

From this and Corollary 2.12, it is not difficult to see that either ζ^λ or $-\zeta^\lambda$ is a simple (super) character, first for λ with $\ell(\lambda)$ even and then for λ with $\ell(\lambda)$ odd.

To show that ζ^λ instead of $-\zeta^\lambda$ is a character of a simple module, it suffices to know that the degree of ζ^λ is positive. The degree formula can be established by induction on $\ell(\lambda)$ (see the proof of [Jo1, Proposition 4.13] for detail). \square

We shall denote by D^λ the irreducible \mathcal{H}_n -module whose character is ζ^λ , for $\lambda \in \mathcal{SP}_n$. The following is an immediate consequence of Theorem 3.3, (3.14), and (3.15).

Proposition 3.7. *Let μ be a composition of d . We have the following decomposition of M^μ as an \mathcal{H}_d -module:*

$$M^\mu \cong \bigoplus_{\lambda \in \mathcal{SP}, \lambda \geq \mu} 2^{\frac{\ell(\lambda)-\delta(\lambda)}{2}} \widehat{K}_{\lambda\mu} D^\lambda,$$

where $\widehat{K}_{\lambda\mu} \in \mathbb{Z}_+$.

4. THE SCHUR-SERGEEV DUALITY

In this section, we formulate a double centralizer property for the actions of the Lie superalgebra $\mathfrak{q}(n)$ and of the algebra \mathcal{H}_d on the tensor superspace $(\mathbb{C}^n|n)^{\otimes d}$. We obtain a multiplicity-free decomposition of $(\mathbb{C}^n|n)^{\otimes d}$ as a $U(\mathfrak{q}(n)) \otimes \mathcal{H}_d$ -module. The characters of the simple $\mathfrak{q}(n)$ -modules arising this way are shown to be Schur Q -functions (up to some 2-powers).

4.1. The classical Schur duality. Let us first recall a general double centralizer property. We reproduce a proof below which can be easily adapted to the superalgebra setting later on.

Proposition 4.1. *Suppose that W is a finite-dimensional vector space, and \mathcal{B} is a semisimple subalgebra of $\text{End}(W)$. Let $\mathcal{A} = \text{End}_{\mathcal{B}}(W)$. Then, $\text{End}_{\mathcal{A}}(W) = \mathcal{B}$.*

As an $\mathcal{A} \otimes \mathcal{B}$ -module, W is multiplicity-free, i.e.,

$$W \cong \bigoplus_i U_i \otimes V_i,$$

where $\{U_i\}$ are pairwise non-isomorphic simple \mathcal{A} -modules and $\{V_i\}$ are pairwise non-isomorphic simple \mathcal{B} -modules.

Proof. Assume that V_a are all the pairwise non-isomorphic simple \mathcal{B} -modules. Then the Hom-spaces $U_a := \text{Hom}_{\mathcal{B}}(V_a, W)$ are naturally \mathcal{A} -modules. By the semisimplicity assumption on \mathcal{B} , we have a \mathcal{B} -module isomorphism:

$$W \cong \bigoplus_a U_a \otimes V_a.$$

By applying Schur's Lemma, we obtain

$$\mathcal{A} = \text{End}_{\mathcal{B}}(W) \cong \bigoplus_a \text{End}_{\mathcal{B}}(U_a \otimes V_a) \cong \bigoplus_a \text{End}(U_a) \otimes \text{id}_{V_a}.$$

Hence \mathcal{A} is semisimple and U_a are all the pairwise non-isomorphic simple \mathcal{A} -modules.

Since \mathcal{A} is now semisimple, we can reverse the roles of \mathcal{A} and \mathcal{B} in the above computation of $\text{End}_{\mathcal{B}}(W)$, and obtain the following isomorphism:

$$\text{End}_{\mathcal{A}}(W) \cong \bigoplus_a \text{id}_{U_a} \otimes \text{End}(V_a) \cong \mathcal{B}.$$

The proposition is proved. \square

The natural action of $\mathfrak{gl}(n)$ on \mathbb{C}^n induces a representation $(\omega_d, (\mathbb{C}^n)^{\otimes d})$ of the general linear Lie algebra $\mathfrak{gl}(n)$, and we have a representation $(\psi_d, (\mathbb{C}^n)^{\otimes d})$ of the symmetric group \mathfrak{S}_d by permutations of the tensor factors.

Theorem 4.2 (Schur duality). *The images $\omega_d(U(\mathfrak{gl}(n)))$ and $\psi_d(\mathbb{C}\mathfrak{S}_d)$ satisfy the double centralizer property, i.e.,*

$$\begin{aligned} \omega_d(U(\mathfrak{gl}(n))) &= \text{End}_{\mathbb{C}\mathfrak{S}_d}((\mathbb{C}^n)^{\otimes d}), \\ \text{End}_{\mathfrak{gl}(n)}((\mathbb{C}^n)^{\otimes d}) &= \psi_d(\mathbb{C}\mathfrak{S}_d). \end{aligned}$$

Moreover, as a $\mathfrak{gl}(n) \times \mathfrak{S}_d$ -module,

$$(4.1) \quad (\mathbb{C}^n)^{\otimes d} \cong \bigoplus_{\lambda \in \mathcal{P}_d, \ell(\lambda) \leq n} L(\lambda) \otimes S^\lambda,$$

where $L(\lambda)$ denotes the irreducible $\mathfrak{gl}(n)$ -module of highest weight λ .

We will skip the proof of the Schur duality here, as it is similar to a detailed proof below for its super analogue (Theorems 4.7 and 4.8).

As an application of the Schur duality, let us derive the character formula for $\text{ch} L(\lambda) = \text{tr} x_1^{E_{11}} x_2^{E_{22}} \cdots x_n^{E_{nn}}|_{L(\lambda)}$, where as usual E_{ii} denotes the matrix whose (i, i) th entry is 1 and zero else.

Denote by $\mathcal{CP}_d(n)$ the set of compositions of d of length $\leq n$. Set $W = (\mathbb{C}^n)^{\otimes d}$. Given $\mu \in \mathcal{CP}_d(n)$, let W_μ indicate the μ -weight space of W . Observe that W_μ has a linear basis

$$(4.2) \quad e_{i_1} \otimes \cdots \otimes e_{i_d}, \text{ with } \{i_1, \dots, i_d\} = \underbrace{\{1, \dots, 1\}}_{\mu_1}, \dots, \underbrace{\{n, \dots, n\}}_{\mu_n}.$$

On the other hand, \mathfrak{S}_n acts on the basis (4.2) of W_μ transitively, and the stablizer of the basis element $e_1^{\mu_1} \otimes \cdots \otimes e_n^{\mu_n}$ is the Young subgroup \mathfrak{S}_μ . Therefore we have $W_\mu \cong \text{Ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_d} \mathbf{1}_d$ and hence

$$(4.3) \quad W \cong \bigoplus_{\mu \in \mathcal{CP}_d(n)} W_\mu \cong \bigoplus_{\mu \in \mathcal{CP}_d(n)} \text{Ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_d} \mathbf{1}_d.$$

This and (4.1) imply that

$$\bigoplus_{\mu \in \mathcal{CP}_d(n)} \text{Ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_d} \mathbf{1}_d \cong \bigoplus_{\lambda \in \mathcal{P}_d, \ell(\lambda) \leq n} L(\lambda) \otimes S^\lambda.$$

Applying the trace operator $\text{tr} x_1^{E_{11}} x_2^{E_{22}} \cdots x_n^{E_{nn}}$ and the Frobenius characteristic map ch to both sides of the above isomorphism and summing over d , we obtain

$$\sum_{\mu \in \mathcal{P}, \ell(\mu) \leq n} m_\mu(x_1, \dots, x_n) h_\mu(z) = \sum_{\lambda \in \mathcal{P}, \ell(\lambda) \leq n} \text{ch} L(\lambda) s_\lambda(z),$$

where $z = \{z_1, z_2, \dots\}$ is infinite. Then using the Cauchy identity (3.3) and noting the linear independence of the $s_\lambda(z)$'s, we recover the following well-known character formula:

$$(4.4) \quad \text{ch} L(\lambda) = s_\lambda(x_1, x_2, \dots, x_n).$$

4.2. The queer Lie superalgebras. The associative superalgebra $Q(n)$ (defined in Section 2.2) equipped with the super-commutator is called the *queer Lie superalgebra* and denoted by $\mathfrak{q}(n)$. Let

$$I(n|n) = \{\bar{1}, \dots, \bar{n}, 1, \dots, n\}.$$

The $\mathfrak{q}(n)$ can be explicitly realized as matrices in the $n|n$ block form, indexed by $I(n|n)$:

$$(4.5) \quad \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where a and b are arbitrary $n \times n$ matrices. The even (respectively, odd) part \mathfrak{g}_0 (respectively, \mathfrak{g}_1) of $\mathfrak{g} = \mathfrak{q}(n)$ consists of those matrices of the form (4.5) with $b = 0$ (respectively, $a = 0$). Denote by E_{ij} for $i, j \in I(n|n)$ the standard elementary matrix with the (i, j) th entry being 1 and zero elsewhere.

The standard Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ of \mathfrak{g} consists of matrices of the form (4.5) with a, b being arbitrary diagonal matrices. Noting that $[\mathfrak{h}_0, \mathfrak{h}] = 0$ and $[\mathfrak{h}_1, \mathfrak{h}_1] = \mathfrak{h}_0$, the Lie superalgebra \mathfrak{h} is not abelian. The vectors

$$H_i := E_{\bar{i}, \bar{i}} + E_{ii}, \quad i = 1, \dots, n,$$

is a basis for the \mathfrak{h}_0 . We let $\{\epsilon_i | i = 1, \dots, n\}$ denote the corresponding dual basis in \mathfrak{h}_0^* . With respect to \mathfrak{h}_0 we have the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ with roots $\{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n\}$. For each root α we have $\dim_{\mathbb{C}}(\mathfrak{g}_\alpha)_i = 1$, for $i \in \mathbb{Z}_2$. The system of positive roots corresponding to the Borel subalgebra \mathfrak{b} consisting of matrices of the form (4.5) with a, b upper triangular is given by $\{\epsilon_i - \epsilon_j | 1 \leq i < j \leq n\}$.

The Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ is a solvable Lie superalgebra, and its irreducible representations are described as follows. Let $\lambda \in \mathfrak{h}_0^*$ and consider the symmetric bilinear form $\langle \cdot, \cdot \rangle_\lambda$ on \mathfrak{h}_1 defined by

$$\langle v, w \rangle_\lambda := \lambda([v, w]).$$

Denote by $\text{Rad}\langle \cdot, \cdot \rangle_\lambda$ the radical of the form $\langle \cdot, \cdot \rangle_\lambda$. Then the form $\langle \cdot, \cdot \rangle_\lambda$ descends to a nondegenerate symmetric bilinear form on $\mathfrak{h}_1 / \text{Rad}\langle \cdot, \cdot \rangle_\lambda$, and it gives rise to a Clifford superalgebra $\mathcal{C}\ell_\lambda := \mathcal{C}\ell(\mathfrak{h}_1 / \text{Rad}\langle \cdot, \cdot \rangle_\lambda)$. By definition we have an isomorphism of superalgebras

$$\mathcal{C}\ell_\lambda \cong U(\mathfrak{h}) / I_\lambda,$$

where I_λ denotes the ideal of $U(\mathfrak{h})$ generated by $\text{Rad}\langle \cdot, \cdot \rangle_\lambda$ and $a - \lambda(a)$ for $a \in \mathfrak{h}_0$.

Let $\mathfrak{h}'_1 \subseteq \mathfrak{h}_1$ be a maximal isotropic subspace and consider the subalgebra $\mathfrak{h}' = \mathfrak{h}_0 \oplus \mathfrak{h}'_1$. Clearly the one-dimensional \mathfrak{h}_0 -module $\mathbb{C}v_\lambda$, defined by $hv_\lambda = \lambda(h)v_\lambda$, extends trivially to \mathfrak{h}' . Set

$$W_\lambda := \text{Ind}_{\mathfrak{h}'}^{\mathfrak{h}} \mathbb{C}v_\lambda.$$

We see that the action of \mathfrak{h} factors through $\mathcal{C}\ell_\lambda$ so that W_λ becomes the unique irreducible $\mathcal{C}\ell_\lambda$ -module and hence is independent of the choice of \mathfrak{h}'_1 . The following can now be easily verified.

Lemma 4.3. *For $\lambda \in \mathfrak{h}_0^*$, W_λ is an irreducible \mathfrak{h} -module. Furthermore, every finite-dimensional irreducible \mathfrak{h} -module is isomorphic to some W_λ .*

Let V be a finite-dimensional irreducible \mathfrak{g} -module and let W_μ be an irreducible \mathfrak{h} -submodule of V . For every $v \in W_\mu$ we have $hv = \mu(h)v$, for all $h \in \mathfrak{h}_0$. Let α be a positive root with associated root vectors e_α and \bar{e}_α in \mathfrak{n}^+ satisfying $\text{dege}_\alpha = \bar{0}$ and $\text{deg}\bar{e}_\alpha = \bar{1}$. Then the space $\mathbb{C}e_\alpha W_\mu + \mathbb{C}\bar{e}_\alpha W_\mu$ is an \mathfrak{h} -module on which \mathfrak{h}_0 transforms by the character $\mu + \alpha$. Thus by the finite dimensionality of V there exists $\lambda \in \mathfrak{h}_0^*$ and an irreducible \mathfrak{h} -module $W_\lambda \subseteq V$ such that $\mathfrak{n}^+ W_\lambda = 0$. By the irreducibility of V we must have $U(\mathfrak{n}^-)W_\lambda = V$, which gives rise to a weight space decomposition of $V = \bigoplus_{\mu \in \mathfrak{h}_0^*} V_\mu$. The space $W_\lambda = V_\lambda$ is the *highest weight space* of V , and it completely determines the irreducible module V . We denote V by $V(\lambda)$.

Let $\ell(\lambda)$ be the dimension of space $\mathfrak{h}_{\bar{1}}/\text{Rad}\langle \cdot, \cdot \rangle_\lambda$, which equals the number of i such that $\lambda(H_i) \neq 0$. Then the highest weight space W_λ of $V(\lambda)$ has dimension $2^{(\ell(\lambda)+\delta(\lambda))/2}$. It is easy to see that the \mathfrak{h} -module W_λ has an odd automorphism if and only if $\ell(\lambda)$ is an odd integer. An automorphism of the irreducible \mathfrak{g} -module $V(\lambda)$ clearly induces an \mathfrak{h} -module automorphism of its highest weight space. Conversely, any \mathfrak{h} -module automorphism on W_λ induces an automorphism of the \mathfrak{g} -module $\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} W_\lambda$. Since an automorphism preserves the maximal submodule, it induces an automorphism of the unique irreducible quotient \mathfrak{g} -module. Summarizing, we have established the following.

Lemma 4.4. *Let $\mathfrak{g} = \mathfrak{q}(n)$, and \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . Let $\lambda \in \mathfrak{h}_0^*$ and $V(\lambda)$ be an irreducible \mathfrak{g} -module of highest weight λ . We have*

$$\dim \text{End}_{\mathfrak{g}}(V(\lambda)) = \begin{cases} 1, & \text{if } \ell(\lambda) \text{ is even,} \\ 2, & \text{if } \ell(\lambda) \text{ is odd.} \end{cases}$$

4.3. The Sergeev duality. In this subsection, we give a detailed exposition (also see [CW, Chapter 3]) on the results of Sergeev [Se1].

Set $V = \mathbb{C}^{n|n}$. We have a representation $(\Omega_d, V^{\otimes d})$ of $\mathfrak{gl}(n|n)$, hence of its subalgebra $\mathfrak{q}(n)$, and we also have a representation $(\Psi_d, V^{\otimes d})$ of the symmetric group \mathfrak{S}_d defined by

$$\Psi_d(s_i).(v_1 \otimes \dots \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_d) = (-1)^{|v_i| \cdot |v_{i+1}|} v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_d,$$

where $s_i = (i, i+1)$ is the simple reflection and $v_i, v_{i+1} \in V$ are \mathbb{Z}_2 -homogeneous. Moreover, the actions of $\mathfrak{gl}(n|n)$ and the symmetric group \mathfrak{S}_d on $V^{\otimes d}$ commute with each other. Note in addition that the Clifford algebra \mathcal{Cl}_d acts on $V^{\otimes d}$, also denoted by Ψ_d :

$$\Psi_d(c_i).(v_1 \otimes \dots \otimes v_d) = (-1)^{(|v_1| + \dots + |v_{i-1}|)} v_1 \otimes \dots \otimes v_{i-1} \otimes P v_i \otimes \dots \otimes v_d,$$

where $v_i \in V$ is assumed to be \mathbb{Z}_2 -homogeneous and $1 \leq i \leq n$.

Lemma 4.5. *Let $V = \mathbb{C}^{n|n}$. The actions of \mathfrak{S}_d and \mathcal{Cl}_d above give rise to a representation $(\Psi_d, V^{\otimes d})$ of \mathcal{H}_d . Moreover, the actions of $\mathfrak{q}(n)$ and \mathcal{H}_d on $V^{\otimes d}$ super-commute with each other.*

Symbolically, we write

$$\mathfrak{q}(n) \overset{\Omega_d}{\curvearrowright} V^{\otimes d} \overset{\Psi_d}{\curvearrowright} \mathcal{H}_d.$$

Proof. It is straightforward to check that the actions of \mathfrak{S}_d and \mathcal{Cl}_d on $V^{\otimes d}$ are compatible and they give rise to an action of \mathcal{H}_d . By the definition of $\mathfrak{q}(n)$ and the definition of $\Psi_d(c_i)$ via P , the action of $\mathfrak{q}(n)$ (super)commutes with the action of c_i for $1 \leq i \leq d$. Since $\mathfrak{gl}(n|n)$ (super)commutes with \mathfrak{S}_d , so does the subalgebra $\mathfrak{q}(n)$ of $\mathfrak{gl}(n|n)$. Hence, the action of $\mathfrak{q}(n)$ commutes with the action of \mathcal{H}_d on $V^{\otimes d}$. \square

Let us digress on the double centralizer property for superalgebras in general. Note the superalgebra isomorphism

$$Q(m) \otimes Q(n) \cong M(mn|mn).$$

Hence, as a $Q(m) \otimes Q(n)$ -module, the tensor product $\mathbb{C}^{m|m} \otimes \mathbb{C}^{n|n}$ is a direct sum of two isomorphic copies of a simple module (which is $\cong \mathbb{C}^{mn|mn}$), and we have

$\text{Hom}_{Q(n)}(\mathbb{C}^{n|n}, \mathbb{C}^{mn|mn}) \cong \mathbb{C}^{m|m}$ as a $Q(m)$ -module. Let \mathcal{A} and \mathcal{B} be two semisimple superalgebras. Let M be a simple \mathcal{A} -module of type \mathbf{Q} and let N be a simple \mathcal{B} -module of type \mathbf{Q} . Then, by Lemma 2.5, the $\mathcal{A} \otimes \mathcal{B}$ -supermodule $M \otimes N$ is a direct sum of two isomorphic copies of a simple module $M \circledast N$ of type \mathbf{M} , and we shall write $M \circledast N = 2^{-1}M \otimes N$; Moreover, $\text{Hom}_{\mathcal{B}}(N, 2^{-1}M \otimes N)$ is naturally an \mathcal{A} -module, which is isomorphic to the \mathcal{A} -module M . The usual double centralizer property Proposition 4.1 affords the following superalgebra generalization (with essentially the same proof once we keep in mind the Super Schur's Lemma 2.3).

Proposition 4.6. *Suppose that W is a finite-dimensional vector superspace, and \mathcal{B} is a semisimple subalgebra of $\text{End}(W)$. Let $\mathcal{A} = \text{End}_{\mathcal{B}}(W)$. Then, $\text{End}_{\mathcal{A}}(W) = \mathcal{B}$.*

As an $\mathcal{A} \otimes \mathcal{B}$ -module, W is multiplicity-free, i.e.,

$$W \cong \bigoplus_i 2^{-\delta_i} U_i \otimes V_i,$$

where $\delta_i \in \{0, 1\}$, $\{U_i\}$ are pairwise non-isomorphic simple \mathcal{A} -modules, $\{V_i\}$ are pairwise non-isomorphic simple \mathcal{B} -modules. Moreover, U_i and V_i are of same type, and they are of type \mathbf{M} if and only if $\delta_i = 0$.

Theorem 4.7 (Sergeev duality I). *The images $\Omega_d(U(\mathfrak{q}(n)))$ and $\Psi_d(\mathcal{H}_d)$ satisfy the double centralizer property, i.e.,*

$$\begin{aligned} \Omega_d(U(\mathfrak{q}(n))) &= \text{End}_{\mathcal{H}_d}(V^{\otimes d}), \\ \text{End}_{\mathfrak{q}(n)}(V^{\otimes d}) &= \Psi_d(\mathcal{H}_d). \end{aligned}$$

Proof. Write $\mathfrak{g} = \mathfrak{q}(n)$. We will denote by $Q(V)$ the associative subalgebra of endomorphisms on V which super-commute with the linear operator P . By Lemma 4.5, we have $\Omega_d(U(\mathfrak{g})) \subseteq \text{End}_{\mathcal{H}_d}(V^{\otimes d})$.

We shall proceed to prove that $\Omega_d(U(\mathfrak{g})) \supseteq \text{End}_{\mathcal{H}_d}(V^{\otimes d})$. By examining the actions of $\mathcal{C}l_d$ on $V^{\otimes d}$, we see that the natural isomorphism $\text{End}(V)^{\otimes d} \cong \text{End}(V^{\otimes d})$ allows us to identify $\text{End}_{\mathcal{C}l_d}(V^{\otimes d}) \equiv Q(V)^{\otimes d}$. As we recall $\mathcal{H}_d = \mathcal{C}l_d \rtimes \mathfrak{S}_d$, this further leads to the identification $\text{End}_{\mathcal{H}_d}(V^{\otimes d}) \equiv \text{Sym}^d(Q(V))$, the space of \mathfrak{S}_d -invariants in $Q(V)^{\otimes d}$.

Denote by Y_k , $1 \leq k \leq d$, the \mathbb{C} -span of the supersymmetrization

$$\omega(x_1, \dots, x_k) := \sum_{\sigma \in \mathfrak{S}_d} \sigma.(x_1 \otimes \dots \otimes x_k \otimes 1^{d-k}),$$

for all $x_i \in Q(V)$. Note that $Y_d = \text{Sym}^d(Q(V)) \equiv \text{End}_{\mathcal{H}_d}(V^{\otimes d})$.

Let $\tilde{x} = \Omega(x) = \sum_{i=1}^d 1^{i-1} \otimes x \otimes 1^{d-i}$, for $x \in \mathfrak{g} = Q(V)$, and denote by X_k , $1 \leq k \leq d$, the \mathbb{C} -span of $\tilde{x}_1 \dots \tilde{x}_k$ for all $x_i \in \mathfrak{q}(n)$.

Claim. We have $Y_k \subseteq X_k$ for $1 \leq k \leq d$.

Assuming the claim, we have $\Omega_d(U(\mathfrak{g})) = \text{End}_{\mathcal{H}_d}(V^{\otimes d}) = \text{End}_{\mathcal{B}}(V^{\otimes d})$, for $\mathcal{B} := \Psi_d(\mathcal{H}_d)$. Note that the algebra \mathcal{H}_d , and hence also \mathcal{B} , are semisimple superalgebras, and so the assumption of Proposition 4.6 is satisfied. Therefore, we have $\text{End}_{U(\mathfrak{g})}(V^{\otimes d}) = \Psi_d(\mathcal{H}_d)$.

It remains to prove the Claim by induction on k . The case $k = 1$ holds, thanks to $\omega(x) = (d-1)!\tilde{x}$.

Assume that $Y_{k-1} \subseteq X_{k-1}$. Note that $\omega(x_1, \dots, x_{k-1}) \cdot \tilde{x}_k \in X_k$. On the other hand, we have

$$\begin{aligned} & \omega(x_1, \dots, x_{k-1}) \cdot \tilde{x}_k \\ &= \sum_{\sigma \in \mathfrak{S}_d} \sigma.(x_1 \otimes \dots \otimes x_{k-1} \otimes 1^{d-k+1}) \cdot \tilde{x}_k \\ &= \sum_{j=1}^d \sum_{\sigma \in \mathfrak{S}_d} \sigma. \left((x_1 \otimes \dots \otimes x_{k-1} \otimes 1^{d-k+1}) \cdot (1^{j-1} \otimes x_k \otimes 1^{d-j}) \right), \end{aligned}$$

which can be written as a sum $A_1 + A_2$, where

$$A_1 = \sum_{j=1}^{k-1} \omega(x_1, \dots, x_j x_k, \dots, x_{k-1}) \in Y_{k-1},$$

and

$$\begin{aligned} A_2 &= \sum_{j=k}^d \sum_{\sigma \in \mathfrak{S}_d} \sigma.(x_1 \otimes \dots \otimes x_{k-1} \otimes 1^{j-k} \otimes x_k \otimes 1^{d-j}) \\ &= (d - k + 1) \omega(x_1, \dots, x_{k-1}, x_k). \end{aligned}$$

Note that $A_1 \in X_k$, since $Y_{k-1} \subseteq X_{k-1} \subseteq X_k$. Hence, $A_2 \in X_k$, and so, $Y_k \subseteq X_k$. This proves the claim and hence the theorem. \square

Theorem 4.8 (Sergeev duality II). *Let $V = \mathbb{C}^{n|n}$. As a $U(\mathfrak{q}(n)) \otimes \mathcal{H}_d$ -module, we have*

$$(4.6) \quad V^{\otimes d} \cong \bigoplus_{\lambda \in \mathcal{SP}_d, \ell(\lambda) \leq n} 2^{-\delta(\lambda)} V(\lambda) \otimes D^\lambda.$$

Proof. Let $W = V^{\otimes d}$. It follows from the double centralizer property and the semisimplicity of the superalgebra \mathcal{H}_d that we have a multiplicity-free decomposition of the $(\mathfrak{q}(n), \mathcal{H}_d)$ -module W :

$$W \cong \bigoplus_{\lambda \in \Omega_d(n)} 2^{-\delta(\lambda)} V^{[\lambda]} \otimes D^\lambda,$$

where $V^{[\lambda]}$ is some simple $\mathfrak{q}(n)$ -module associated to λ , whose highest weight (with respect to the standard Borel) is to be determined. Also to be determined is the index set $\Omega_d(n) = \{\lambda \in \mathcal{SP}_d \mid V^{[\lambda]} \neq 0\}$.

We shall identify as usual a weight $\mu = \sum_{i=1}^n \mu_i \varepsilon_i$ occuring in W with a composition $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{CP}_d(n)$. We have the following weight space decomposition:

$$(4.7) \quad W = \bigoplus_{\mu \in \mathcal{CP}_d(n)} W_\mu,$$

where W_μ has a linear basis $e_{i_1} \otimes \dots \otimes e_{i_d}$, with the indices satisfying the following equality of sets:

$$\{|i_1|, \dots, |i_d|\} = \{\underbrace{1, \dots, 1}_{\mu_1}, \dots, \underbrace{n, \dots, n}_{\mu_n}\}.$$

We have an \mathcal{H}_d -module isomorphism:

$$(4.8) \quad W_\mu \cong M^\mu,$$

where we recall M^μ denotes the permutation \mathcal{H}_d -module $M^\mu = \mathcal{H}_d \otimes_{\mathbb{C}\mathfrak{S}_\mu} \mathbf{1}_d$.

It follows by Proposition 3.7 and (4.8) that $V^{[\lambda]} = \bigoplus_{\mu \in \mathcal{CP}_d(n), \mu \leq \lambda} V_\mu^{[\lambda]}$, and hence, $\lambda \in \mathcal{P}_d(n)$ if $V^{[\lambda]} \neq 0$. Among all such μ , clearly λ corresponds to a highest weight. Hence, we conclude that $V^{[\lambda]} = V(\lambda)$, the simple \mathfrak{g} -module of highest weight λ , and that $\mathfrak{Q}_d(n) = \{\lambda \in \mathcal{SP}_d \mid \ell(\lambda) \leq n\}$. This completes the proof of Theorem 4.8. \square

4.4. The irreducible character formula for $\mathfrak{q}(n)$. A character of a $\mathfrak{q}(n)$ -module with weight space decomposition $M = \bigoplus M_\mu$ is defined to be

$$\mathrm{tr} x_1^{H_1} \dots x_n^{H_n} |_M = \sum_{\mu=(\mu_1, \dots, \mu_n)} \dim M_\mu \cdot x_1^{\mu_1} \dots x_n^{\mu_n}.$$

Theorem 4.9. *Let λ be a strict partition of length $\leq n$. The character of the simple $\mathfrak{q}(n)$ -module $V(\lambda)$ is given by*

$$\mathrm{ch} V(\lambda) = 2^{-\frac{\ell(\lambda) - \delta(\lambda)}{2}} Q_\lambda(x_1, \dots, x_n).$$

Proof. By (4.7) and (4.8), we have

$$V^{\otimes d} = \bigoplus_{\mu \in \mathcal{CP}_d(n)} \mathrm{Ind}_{\mathfrak{S}_\mu}^{\mathcal{H}_d} \mathbf{1}_d.$$

Applying ch^- and the trace operator $\mathrm{tr} x_1^{H_1} \dots x_n^{H_n}$ to this decomposition of $V^{\otimes d}$ simultaneously, which we will denote by ch^2 , we obtain that

$$\begin{aligned} \sum_d \mathrm{ch}^2(V^{\otimes d}) &= \sum_{\mu \in \mathcal{P}, \ell(\mu) \leq n} q_\mu(z) m_\mu(x) \\ &= \prod_{1 \leq i \leq n, 1 \leq j} \frac{1 + x_i z_j}{1 - x_i z_j} \\ &= \sum_{\lambda \in \mathcal{SP}} 2^{-\ell(\lambda)} Q_\lambda(x_1, \dots, x_n) Q_\lambda(z), \end{aligned}$$

where the last equation is the Cauchy identity in Theorem 3.4 and the middle equation can be verified directly.

On the other hand, by applying ch^2 to (4.6) and using (3.14), we obtain that

$$\sum_d \mathrm{ch}^2(V^{\otimes d}) = \sum_{\lambda \in \mathcal{SP}, \ell(\lambda) \leq n} 2^{-\delta(\lambda)} \mathrm{ch} V(\lambda) \cdot 2^{-\frac{\ell(\lambda) - \delta(\lambda)}{2}} Q_\lambda(z).$$

Now the theorem follows by comparing the above two identities and noting the linear independence of the $Q_\lambda(z)$'s. \square

5. THE COINVARIANT ALGEBRA AND GENERALIZATIONS

In this section, we formulate a graded regular representation for \mathcal{H}_n , which is a spin analogue of the coinvariant algebra for \mathfrak{S}_n . We also study its generalizations which involve the symmetric algebra $S^*\mathbb{C}^n$ and the exterior algebra $\wedge^*\mathbb{C}^n$. We solve the corresponding graded multiplicity problems in terms of specializations of Schur Q -functions. In addition, a closed formula for the principal specialization $Q_\xi(1, t, t^2, \dots)$ of the Schur Q -function is given.

5.1. A commutative diagram. Recall a homomorphism φ (cf. [Mac, III, §8, Example 10]) defined by

$$(5.1) \quad \begin{aligned} \varphi : \Lambda &\longrightarrow \Gamma, \\ \varphi(p_r) &= \begin{cases} 2p_r, & \text{for } r \text{ odd,} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where p_r denotes the r th power sum. Denote

$$H(t) = \sum_{n \geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t} = \exp \left(\sum_{r \geq 1} \frac{p_r t^r}{r} \right).$$

Note that $Q(t)$ from (3.6) can be rewritten as

$$Q(t) = \exp \left(2 \sum_{r \geq 1, r \text{ odd}} \frac{p_r t^r}{r} \right),$$

and so we see that

$$(5.2) \quad \varphi(H(t)) = Q(t).$$

Hence, we have $\varphi(h_n) = q_n$ for all $n \geq 0$, and

$$(5.3) \quad \varphi(h_\mu) = q_\mu, \quad \forall \mu \in \mathcal{P}.$$

Given an \mathfrak{S}_n -module M , the algebra \mathcal{H}_n acts naturally on $\mathcal{C}l_n \otimes M$, where $\mathcal{C}l_n$ acts by left multiplication on the first factor and \mathfrak{S}_n acts diagonally. We have an isomorphism of \mathcal{H}_n -modules:

$$(5.4) \quad \mathcal{C}l_n \otimes M \cong \text{Ind}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} M.$$

Following [WW2], we define a functor for $n \geq 0$

$$\begin{aligned} \Phi_n : \mathfrak{S}_n\text{-mod} &\longrightarrow \mathcal{H}_n\text{-mod} \\ \Phi_n(M) &= \text{ind}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} M. \end{aligned}$$

Such a sequence $\{\Phi_n\}$ induces a \mathbb{Z} -linear map on the Grothendieck group level:

$$\Phi : R \longrightarrow R^-,$$

by letting $\Phi([M]) = [\Phi_n(M)]$ for $M \in \mathfrak{S}_n\text{-mod}$.

Recall that R carries a natural Hopf algebra structure with multiplication given by induction and comultiplication given by restriction [Ze]. In the same fashion, we can define a Hopf algebra structure on R^- by induction and restriction. On the other hand, $\Lambda_{\mathbb{Q}} \cong \mathbb{Q}[p_1, p_2, p_3, \dots]$ is naturally a Hopf algebra, where each p_r is a primitive element,

and $\Gamma_{\mathbb{Q}} \cong \mathbb{Q}[p_1, p_3, p_5, \dots]$ is naturally a Hopf subalgebra of $\Lambda_{\mathbb{Q}}$. The characteristic map $\text{ch} : R_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ is an isomorphism of Hopf algebras (cf. [Ze]). A similar argument easily shows that the map $\text{ch}^- : R_{\mathbb{Q}}^- \rightarrow \Gamma_{\mathbb{Q}}$ is an isomorphism of Hopf algebras.

Proposition 5.1. [WW2] *The map $\Phi : R_{\mathbb{Q}} \rightarrow R_{\mathbb{Q}}^-$ is a homomorphism of Hopf algebras. Moreover, we have the following commutative diagram of Hopf algebras:*

$$(5.5) \quad \begin{array}{ccc} R_{\mathbb{Q}} & \xrightarrow{\Phi} & R_{\mathbb{Q}}^- \\ \text{ch} \downarrow \cong & & \text{ch}^- \downarrow \cong \\ \Lambda_{\mathbb{Q}} & \xrightarrow{\varphi} & \Gamma_{\mathbb{Q}} \end{array}$$

Proof. Using (3.2) and (5.3) we have

$$\varphi(\text{ch}(\text{ind}_{\mathbb{C}\mathfrak{S}_{\mu}}^{\mathbb{C}\mathfrak{S}_n} \mathbf{1}_n)) = q_{\mu}.$$

On the other hand, it follows by (3.15) that

$$\text{ch}^-(\Phi(\text{ind}_{\mathbb{C}\mathfrak{S}_{\mu}}^{\mathbb{C}\mathfrak{S}_n} \mathbf{1}_n)) = \text{ch}^-(\text{ind}_{\mathbb{C}\mathfrak{S}_{\mu}}^{\mathcal{H}_n} \mathbf{1}_n) = q_{\mu}.$$

This establishes the commutative diagram on the level of linear maps, since R_n has a basis given by the characters of the permutation modules $\text{ind}_{\mathbb{C}\mathfrak{S}_{\mu}}^{\mathbb{C}\mathfrak{S}_n} \mathbf{1}_n$ for $\mu \in \mathcal{P}_n$.

It can be verified easily that $\varphi : \Lambda_{\mathbb{Q}} \rightarrow \Gamma_{\mathbb{Q}}$ is a homomorphism of Hopf algebras. Since both ch and ch^- are isomorphisms of Hopf algebras, it follows from the commutativity of (5.5) that $\Phi : R_{\mathbb{Q}} \rightarrow R_{\mathbb{Q}}^-$ is a homomorphism of Hopf algebras. \square

We shall use the commutation diagram (5.5) as a bridge to discuss spin generalizations of some known constructions in the representation theory of symmetric groups, such as the coinvariant algebras, Kostka polynomials, etc.

5.2. The coinvariant algebra for \mathfrak{S}_n . The symmetric group \mathfrak{S}_n acts on $V = \mathbb{C}^n$ and then on the symmetric algebra S^*V , which is identified with $\mathbb{C}[x_1, \dots, x_n]$ naturally. It is well known that the algebra of \mathfrak{S}_n -invariants on S^*V , or equivalently $\mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}$, is a polynomial algebra in e_1, e_2, \dots, e_n , where $e_i = e_i[x_1, \dots, x_n]$ denotes the i th elementary symmetric polynomial.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n , denote

$$(5.6) \quad n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i.$$

We also denote by $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$ the *hook length* and $c_{ij} = j - i$ the *content* associated to a cell (i, j) in the Young diagram of λ .

Example 5.2. For $\lambda = (4, 3, 1)$, the hook lengths are listed in the corresponding cells as follows:

6	4	3	1
4	2	1	
1			

In this case, $n(\lambda) = 5$.

Denote by $t^\bullet = (1, t, t^2, \dots)$ for a formal variable t . We have the following principal specialization of the r th power-sum:

$$p_r(t^\bullet) = \frac{1}{1 - t^r}.$$

The following well-known formula (cf. [Mac, I, §3, 2]) for the principal specialization of s_λ can be proved in a multiple of ways:

$$(5.7) \quad s_\lambda(t^\bullet) = \frac{t^{n(\lambda)}}{\prod_{(i,j) \in \lambda} (1 - t^{h_{ij}})}.$$

Write formally

$$S_t V = \sum_{j \geq 0} t^j (S^j V).$$

Consider the graded multiplicity of a given Specht module S^λ for a partition λ of n in the graded algebra S^*V , which is by definition

$$f_\lambda(t) := \dim \operatorname{Hom}_{\mathfrak{S}_n}(S^\lambda, S_t V).$$

The *coinvariant algebra* of \mathfrak{S}_n is defined to be

$$(S^*V)_{\mathfrak{S}_n} = S^*V/I,$$

where I denotes the ideal generated by e_1, \dots, e_n . By a classical theorem of Chevalley (cf. [Ka]), we have an isomorphism of \mathfrak{S}_n -modules:

$$(5.8) \quad S^*V \cong (S^*V)_{\mathfrak{S}_n} \otimes (S^*V)^{\mathfrak{S}_n}.$$

Define the polynomial

$$f^\lambda(t) := \dim \operatorname{Hom}_{\mathfrak{S}_n}(S^\lambda, (S_t V)_{\mathfrak{S}_n}).$$

Closed formulas for $f_\lambda(t)$ and $f^\lambda(t)$ in various forms have been well known (cf. Steinberg [S], Lusztig [Lu1], Kirillov [Ki]). Following Lusztig, $f^\lambda(t)$ is called the *fake degree* in connection with Hecke algebras and finite groups of Lie type. We will skip a proof of Theorem 5.3 below, as it can be read off by specializing $s = 0$ in the proof of Theorem 5.4. Thanks to (5.8), the formula (5.10) is equivalent to (5.9).

Theorem 5.3. *The following formulas for the graded multiplicities hold:*

$$(5.9) \quad f_\lambda(t) = \frac{t^{n(\lambda)}}{\prod_{(i,j) \in \lambda} (1 - t^{h_{ij}})},$$

$$(5.10) \quad f^\lambda(t) = \frac{t^{n(\lambda)}(1-t)(1-t^2)\dots(1-t^n)}{\prod_{(i,j) \in \lambda} (1 - t^{h_{ij}})}.$$

Note that the dimension of the Specht module S^λ is given by the hook formula

$$f^\lambda(1) = \frac{n!}{\prod_{(i,j) \in \lambda} h_{ij}}.$$

Setting $t \mapsto 1$ in (5.10) confirms that the coinvariant algebra $(S^*V)_{\mathfrak{S}_n}$ is a regular representation of \mathfrak{S}_n .

5.3. The graded multiplicity in $S^*V \otimes \wedge^*V$ and $S^*V \otimes S^*V$. Recall that $x = \{x_1, x_2, \dots\}$ and $y = \{y_1, y_2, \dots\}$ are two independent sets of variables. Recall a well-known formula relating Schur and skew Schur functions: $s_\lambda(x, y) = \sum_{\rho \subseteq \lambda} s_\rho(x) s_{\lambda/\rho}(y)$. For $\lambda \in \mathcal{P}$, the *super Schur function* (also known as *hook Schur function*) $hs_\lambda(x; y)$ is defined as

$$(5.11) \quad hs_\lambda(x; y) = \sum_{\rho \subseteq \lambda} s_\rho(x) s_{\lambda'/\rho'}(y).$$

In other words, $hs_\lambda(x; y) = \omega_y(s_\lambda(x, y))$, where ω_y is the standard involution on the ring of symmetric functions in y . We refer to [CW, Appendix A] for more detail.

Since $\omega_y(p_r(y)) = (-1)^{r-1} p_r(y)$, $p_r(x; y) := \omega_y(p_r(x, y))$ for $r \geq 1$ is given by

$$p_r(x; y) = \sum_i x_i^r - \sum_j (-y_j)^r.$$

Applying ω_y to the Cauchy identity (3.3) gives us

$$(5.12) \quad \sum_{\lambda \in \mathcal{P}} s_\lambda(z) hs_\lambda(x; y) = \frac{\prod_{j,k} (1 + y_j z_k)}{\prod_{i,k} (1 - x_i z_k)}.$$

Let a, b be variables. The formula in [Mac, Chapter I, §3, 3] can be interpreted as the specialization of $hs_\lambda(x; y)$ at $x = at^\bullet$ and $y = bt^\bullet$:

$$(5.13) \quad hs_\lambda(at^\bullet; bt^\bullet) = t^{n(\lambda)} \prod_{(i,j) \in \lambda} \frac{a + bt^{c_{ij}}}{1 - t^{h_{ij}}}.$$

The \mathfrak{S}_n -action on $V = \mathbb{C}^n$ induces a natural \mathfrak{S}_n -action on the exterior algebra

$$\wedge^* V = \bigoplus_{j=0}^n \wedge^j V.$$

This gives rise to a $\mathbb{Z}_+ \times \mathbb{Z}_+$ bi-graded $\mathbb{C}\mathfrak{S}_n$ -module structure on

$$S^*V \otimes \wedge^*V = \bigoplus_{i \geq 0, 0 \leq j \leq n} S^i V \otimes \wedge^j V.$$

Let s be a variable and write formally

$$\wedge_s V = \sum_{j=0}^n s^j (\wedge^j V).$$

Let $\widehat{f}_\lambda(t, s)$ be the bi-graded multiplicity of the Specht module S^λ for $\lambda \in \mathcal{P}_n$ in $S^*V \otimes \wedge^*V$, which is by definition

$$\widehat{f}_\lambda(t, s) = \dim \operatorname{Hom}_{\mathfrak{S}_n}(S^\lambda, S_t V \otimes \wedge_s V).$$

Theorem 5.4. *Suppose $\lambda \in \mathcal{P}_n$. Then*

- (1) $\widehat{f}_\lambda(t, s) = hs_\lambda(t^\bullet; st^\bullet)$.
- (2) $\widehat{f}_\lambda(t, s) = \frac{t^{n(\lambda)} \prod_{(i,j) \in \lambda} (1 + st^{c_{ij}})}{\prod_{(i,j) \in \lambda} (1 - t^{h_{ij}})} = \frac{\prod_{(i,j) \in \lambda} (t^{i-1} + st^{j-1})}{\prod_{(i,j) \in \lambda} (1 - t^{h_{ij}})}.$

Formula (2) above in the second expression for $\widehat{f}_\lambda(t, s)$ was originally established with a bijective proof by Kirillov-Pak [KP], with $(t^i + st^j)$ being corrected as $(t^{i-1} + st^{j-1})$ above. Our proof below is different, making clear the connection with the specialization of super Schur functions.

Proof. It suffices to prove (1), as (2) follows from (5.13) and (1).

By the definition of $\widehat{f}_\lambda(t, s)$ and the characteristic map, we have

$$(5.14) \quad \text{ch}(S_t V \otimes \wedge_s V) = \widehat{f}_\lambda(t, s) s_\lambda(z).$$

Take $\sigma = (1, 2, \dots, \mu_1)(\mu_1 + 1, \dots, \mu_1 + \mu_2) \cdots$ in \mathfrak{S}_n of type $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ with $\ell = \ell(\mu)$. Note that σ permutes the monomial basis for S^*V , and the monomials fixed by σ are of the form

$$(x_1 x_2 \dots x_{\mu_1})^{a_1} (x_{\mu_1+1} \dots x_{\mu_1+\mu_2})^{a_2} \dots (x_{\mu_1+\dots+\mu_{\ell-1}+1} \dots x_n)^{a_\ell},$$

where $a_1, \dots, a_\ell \in \mathbb{Z}_+$. Let us denote by dx_1, \dots, dx_n the generators for \wedge^*V . Similarly, the exterior monomials fixed by σ up to a sign are of the form

$$(dx_1 dx_2 \dots dx_{\mu_1})^{b_1} (dx_{\mu_1+1} \dots dx_{\mu_1+\mu_2})^{b_2} \dots (dx_{\mu_1+\dots+\mu_{\ell-1}+1} \dots dx_n)^{b_\ell},$$

where $b_1, \dots, b_\ell \in \{0, 1\}$. The sign here is $(-1)^{\sum_i b_i(\mu_i-1)}$.

From these we deduce that

$$\begin{aligned} \text{tr} \sigma|_{S_t V \otimes \wedge_s V} &= \sum_{a_1, \dots, a_\ell \geq 0, (b_1, \dots, b_\ell) \in \mathbb{Z}_2^n} t^{\sum_{i=1}^\ell a_i \mu_i} s^{\sum_{i=1}^\ell b_i \mu_i} (-1)^{\sum_i b_i(\mu_i-1)} \\ &= \frac{(1 - (-s)^{\mu_1})(1 - (-s)^{\mu_2}) \dots (1 - (-s)^{\mu_\ell})}{(1 - t^{\mu_1})(1 - t^{\mu_2}) \dots (1 - t^{\mu_\ell})}. \end{aligned}$$

We shall denote $[u^n]g(u)$ the coefficient of u^n in a power series expansion of $g(u)$. Applying the characteristic map ch , we obtain that

$$\begin{aligned} (5.15) \quad \text{ch}(S_t V \otimes \wedge_s V) &= \sum_{\mu \in \mathcal{P}_n} z_\mu^{-1} \frac{(1 - (-s)^{\mu_1})(1 - (-s)^{\mu_2}) \dots (1 - (-s)^{\mu_\ell})}{(1 - t^{\mu_1})(1 - t^{\mu_2}) \dots (1 - t^{\mu_\ell})} p_\mu \\ &= [u^n] \sum_{\mu \in \mathcal{P}} z_\mu^{-1} u^{|\mu|} p_\mu(t^\bullet; st^\bullet) p_\mu \\ &= [u^n] \prod_{j \geq 0} \prod_i \frac{1 + ust^j z_i}{1 - ut^j z_i} \\ &= \sum_{\lambda \in \mathcal{P}_n} h s_\lambda(t^\bullet; st^\bullet) s_\lambda(z), \end{aligned}$$

where the last equation used the Cauchy identity (5.12). By comparing (5.14) and (5.15), we have proved (1). \square

We can also consider the bi-graded multiplicity of Specht modules S^λ for $\lambda \in \mathcal{P}_n$ in the $\mathbb{C}\mathfrak{S}_n$ -module $S^*V \otimes S^*V$, which by definition is

$$\widetilde{f}_\lambda(t, s) = \dim \text{Hom}_{\mathfrak{S}_n}(S^\lambda, S_t V \otimes S_s V).$$

Theorem 5.5. [BL] *We have $\tilde{f}_\lambda(t, s) = s_\lambda(t^\bullet s^\bullet)$, for $\lambda \in \mathcal{P}$, where $t^\bullet s^\bullet$ indicates the variables $\{t^j s^k \mid j, k \geq 0\}$.*

Proof. By the definition of $\tilde{f}_\lambda(t, s)$, we have

$$(5.16) \quad \text{ch}(S_t V \otimes S_s V) = \tilde{f}_\lambda(t, s) s_\lambda(z).$$

Arguing similarly as in the proof of Theorem 5.4, one deduces that

$$(5.17) \quad \begin{aligned} \text{ch}(S_t V \otimes S_s V) &= \sum_{\mu \in \mathcal{P}_n} z_\mu^{-1} \frac{1}{(1-t^{\mu_1})(1-t^{\mu_2}) \dots (1-t^{\mu_\ell})} \cdot \frac{1}{(1-s^{\mu_1})(1-s^{\mu_2}) \dots (1-s^{\mu_\ell})} p_\mu \\ &= [u^n] \sum_{\mu \in \mathcal{P}} z_\mu^{-1} u^{|\mu|} p_\mu(t^\bullet s^\bullet) p_\mu \\ &= [u^n] \prod_{j,k \geq 0} \prod_i \frac{1}{1 - ut^j s^k z_i} \\ &= \sum_{\lambda \in \mathcal{P}_n} s_\lambda(t^\bullet s^\bullet) s_\lambda(z), \end{aligned}$$

where the last equation used the Cauchy identity (3.3). The theorem is proved by comparing (5.16) and (5.17). \square

Remark 5.6. By (5.8) and Theorem 5.5, the graded multiplicity of S^λ for $\lambda \in \mathcal{P}_n$ in the \mathfrak{S}_n -module $(S^* V)_{\mathfrak{S}_n} \otimes (S^* V)_{\mathfrak{S}_n}$ is $\prod_{r=1}^n (1-t^r)(1-s^r) s_\lambda(t^\bullet s^\bullet)$. This recovers Bergeron-Lamontagne [BL, Theorem 6.1 or (6.4)].

5.4. The spin coinvariant algebra for \mathcal{H}_n . Suppose that the main diagonal of the Young diagram λ contains r cells. Let $\alpha_i = \lambda_i - i$ be the number of cells in the i th row of λ strictly to the right of (i, i) , and let $\beta_i = \lambda'_i - i$ be the number of cells in the i th column of λ strictly below (i, i) , for $1 \leq i \leq r$. We have $\alpha_1 > \alpha_2 > \dots > \alpha_r \geq 0$ and $\beta_1 > \beta_2 > \dots > \beta_r \geq 0$. Then the Frobenius notation for a partition is $\lambda = (\alpha_1, \dots, \alpha_r \mid \beta_1, \dots, \beta_r)$. For example, if $\lambda = (5, 4, 3, 1)$ whose corresponding Young diagram is

$$\lambda = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \\ \hline \square & \square & \square & & \\ \hline \square & & & & \\ \hline \end{array}$$

then $\alpha = (4, 2, 0)$, $\beta = (3, 1, 0)$ and hence $\lambda = (4, 2, 0 \mid 3, 1, 0)$ in Frobenius notation.

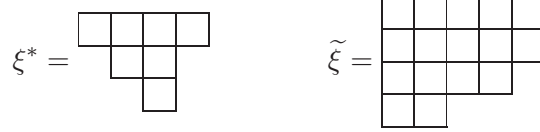
Suppose that ξ is a strict partition of n . Let ξ^* be the associated shifted diagram, that is,

$$\xi^* = \{(i, j) \mid 1 \leq i \leq l(\lambda), i \leq j \leq \lambda_i + i - 1\}$$

which is obtained from the ordinary Young diagram by shifting the k th row to the right by $k - 1$ squares, for each k . Denoting $\ell(\xi) = \ell$, we define the *double partition* ξ to be $\tilde{\xi} = (\xi_1, \dots, \xi_\ell \mid \xi_1 - 1, \xi_2 - 1, \dots, \xi_\ell - 1)$ in Frobenius notation. Clearly, the shifted diagram ξ^* coincides with the part of $\tilde{\xi}$ that lies above the main diagonal. For each cell

$(i, j) \in \xi^*$, denote by h_{ij}^* the associated hook length in the Young diagram $\tilde{\xi}$, and set the content $c_{ij} = j - i$.

Example 5.7. Let $\xi = (4, 3, 1)$. The corresponding shifted diagram ξ^* and double diagram $\tilde{\xi}$ are



The contents of ξ are listed in the corresponding cell of ξ^* as follows:

0	1	2	3
	0	1	2
		0	

The shifted hook lengths for each cell in ξ^* are the usual hook lengths for the corresponding cell in ξ^* , as part of the double diagram $\tilde{\xi}$, as follows:

	7	5	4	2
		4	3	1
			1	

7	5	4	2
	4	3	1
		1	

Since $(S^*V)_{\mathfrak{S}_n}$ is a regular representation of \mathfrak{S}_n , $\mathcal{C}l_n \otimes (S^*V)_{\mathfrak{S}_n}$ is a regular representation of \mathcal{H}_n by (5.4). Denote by

$$d_{\xi}(t) = \dim \operatorname{Hom}_{\mathcal{H}_n}(D^{\xi}, \mathcal{C}l_n \otimes S_t V),$$

$$d^{\xi}(t) = \dim \operatorname{Hom}_{\mathcal{H}_n}(D^{\xi}, \mathcal{C}l_n \otimes (S_t V)_{\mathfrak{S}_n}).$$

The polynomial $d^{\xi}(t)$ will be referred to as the *spin fake degree* of the simple \mathcal{H}_n -module D^{ξ} , and it specializes to the degree of D^{ξ} as t goes to 1. Note $d^{\xi}(t) = d_{\xi}(t) \prod_{r=1}^n (1 - t^r)$.

Theorem 5.8. [WW1] *Let ξ be a strict partition of n . Then*

$$(1) \quad d_{\xi}(t) = 2^{-\frac{\ell(\xi) - \delta(\xi)}{2}} Q_{\xi}(t^{\bullet}).$$

$$(2) \quad d^{\xi}(t) = 2^{-\frac{\ell(\xi) - \delta(\xi)}{2}} t^{n(\xi)} \frac{\prod_{r=1}^n (1 - t^r) \prod_{(i,j) \in \xi^*} (1 + t^{c_{ij}})}{\prod_{(i,j) \in \xi^*} (1 - t^{h_{ij}^*})}.$$

Proof. Let us first prove (1). By Lemma 3.1, the value of the character ξ^n of the basic spin \mathcal{H}_n -module at an element $\sigma \in \mathfrak{S}_n$ of cycle type $\mu \in \mathcal{OP}_n$ is $\xi_{\mu}^n = 2^{\ell(\mu)}$. When combining with the computation in the proof of Theorem 5.4, we have

$$\operatorname{tr} \sigma|_{\mathcal{C}l_n \otimes S_t V} = \frac{2^{\ell(\mu)}}{(1 - t^{\mu_1})(1 - t^{\mu_2}) \dots (1 - t^{\mu_{\ell}})}.$$

Applying the characteristic map $\text{ch}^- : R^- \rightarrow \Gamma_{\mathbb{Q}}$, we obtain that

$$\begin{aligned}
 (5.18) \quad \text{ch}^-(\mathcal{C}l_n \otimes S_t V) &= \sum_{\mu \in \mathcal{OP}_n} z_{\mu}^{-1} \frac{2^{\ell(\mu)}}{(1-t^{\mu_1})(1-t^{\mu_2}) \dots (1-t^{\mu_{\ell}})} p_{\mu} \\
 &= [u^n] \sum_{\mu \in \mathcal{OP}} 2^{\ell(\mu)} z_{\mu}^{-1} u^{|\mu|} p_{\mu}(t^{\bullet}) p_{\mu} \\
 &= [u^n] \prod_{m \geq 0} \prod_i \frac{1 + ut^m z_i}{1 - ut^m z_i} \\
 &= \sum_{\lambda \in \mathcal{SP}_n} 2^{-\ell(\lambda)} Q_{\xi}(t^{\bullet}) Q_{\xi}(z),
 \end{aligned}$$

where the last two equations used (3.10) and the Cauchy identity from Theorem 3.4. It also follows by (3.14) and the definition of $d_{\xi}(t)$ that

$$\text{ch}^-(\mathcal{C}l_n \otimes S_t V) = \sum_{\xi \in \mathcal{SP}_n} 2^{-\frac{\ell(\xi) + \delta(\xi)}{2}} d_{\xi}(t) Q_{\xi}(z).$$

Comparing these two different expressions for $\text{ch}^-(\mathcal{C}l_n \otimes S_t V)$ and noting the linear independence of $Q_{\xi}(z)$, we have proved (1). Part (2) follows by (1) and applying Theorem 5.9 below. \square

Theorem 5.9. *The following holds for any $\xi \in \mathcal{SP}$:*

$$Q_{\xi}(t^{\bullet}) = t^{n(\xi)} \prod_{(i,j) \in \xi^*} \frac{1 + t^{c_{ij}}}{1 - t^{h_{ij}^*}} = \prod_{(i,j) \in \xi^*} \frac{t^{i-1} + t^{j-1}}{1 - t^{h_{ij}^*}}.$$

Theorem 5.9 in a different form was proved by Rosengren [Ro] using formal Schur function identities, and in the current form was proved in [WW1, Section 2] by setting up a bijection between marked shifted tableaux and new combinatorial objects called colored shifted tableaux. The following new proof follows an approach suggested by a referee of [WW1].

Proof. Recall the homomorphism $\varphi : \Lambda \rightarrow \Gamma$ from (5.1). For $\lambda \in \mathcal{P}$, let $S_{\lambda} \in \Gamma$ be the determinant (cf. [Mac, III, §8, 7(a)])

$$S_{\lambda} = \det(q_{\lambda_i - i + j}).$$

It follows by the Jacobi-Trudi identity for s_{λ} and (5.3) that

$$(5.19) \quad \varphi(s_{\lambda}) = S_{\lambda}.$$

Applying φ to the Cauchy identity (3.3) and using (5.2) with $t = z_i$, we obtain that

$$(5.20) \quad \prod_{i,j \geq 1} \frac{1 + x_i z_j}{1 - x_i z_j} = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(z) S_{\lambda}(x).$$

This together with (5.12) implies that

$$(5.21) \quad S_{\lambda}(x) = h s_{\lambda}(x; x).$$

Recall the definition of the double diagram $\tilde{\xi}$ from Section 5.4. It follows from [You, Theorem 3] (cf. [Mac, III, §8, 10]) that

$$\varphi(s_{\tilde{\xi}}) = 2^{-\ell(\xi)} Q_{\tilde{\xi}}^2, \quad \forall \xi \in \mathcal{SP}_n,$$

and hence by (5.19) we have

$$(5.22) \quad Q_{\tilde{\xi}}^2 = 2^{\ell(\xi)} S_{\tilde{\xi}}, \quad \forall \xi \in \mathcal{SP}_n.$$

By (5.13) and (5.21), we have

$$(5.23) \quad S_{\tilde{\xi}}(t^{\bullet}) = t^{n(\tilde{\xi})} \prod_{(i,j) \in \tilde{\xi}} \frac{1 + t^{c_{ij}}}{1 - t^{h_{ij}}} = \prod_{(i,j) \in \tilde{\xi}} \frac{t^{i-1} + t^{j-1}}{1 - t^{h_{ij}}}.$$

Let $\ell = \ell(\xi)$. Denote by H_r the r th hook which consists of the cells below or to the right of a given cell (r, r) on the diagonal of $\tilde{\xi}$ (including (r, r)). For a fixed r , we have

$$\begin{aligned} \prod_{(i,j) \in H_r} (t^{i-1} + t^{j-1}) &= \frac{(t^{r-1} + t^{\xi_r + r - 1})}{t^{r-1} + t^{r-1}} \prod_{r \leq j \leq \xi_r + r - 1} (t^{r-1} + t^{j-1})^2 \\ &= \frac{1 + t^{\xi_r}}{2} \prod_{(r,j) \in \xi^*} (t^{r-1} + t^{j-1})^2. \end{aligned}$$

Hence,

$$\begin{aligned} (5.24) \quad \prod_{(i,j) \in \tilde{\xi}} (t^{i-1} + t^{j-1}) &= \prod_{1 \leq r \leq \ell} \prod_{(i,j) \in H_r} (t^{i-1} + t^{j-1}) \\ &= 2^{-\ell} \prod_{r=1}^{\ell} (1 + t^{\xi_r}) \prod_{(i,j) \in \xi^*} (t^{i-1} + t^{j-1})^2. \end{aligned}$$

On the other hand, for a fixed i , the hook lengths h_{ij} for $(i, j) \in \tilde{\xi}$ with $j > i$ are exactly the hook lengths h_{ij}^* for $(i, j) \in \xi^*$, which are $1, 2, \dots, \xi_i, \xi_i + \xi_{i+1}, \xi_i + \xi_{i+2}, \dots, \xi_i + \xi_{\ell}$ with exception $\xi_i - \xi_{i+1}, \xi_i - \xi_{i+2}, \dots, \xi_i - \xi_{\ell}$ (cf. [Mac, III, §8, 12]). Meanwhile, one can deduce that the hook lengths h_{ki} for $(k, i) \in \tilde{\xi}$ with $k \geq i$ for a given i are $1, 2, \dots, \xi_i - 1, 2\xi_i, \xi_i + \xi_{i+1}, \xi_i + \xi_{i+2}, \dots, \xi_i + \xi_{\ell}$ with exception $\xi_i - \xi_{i+1}, \xi_i - \xi_{i+2}, \dots, \xi_i - \xi_{\ell}$. This means

$$(5.25) \quad \prod_{(i,j) \in \tilde{\xi}} (1 - t^{h_{ij}}) = \prod_{(i,j) \in \xi^*} (1 - t^{h_{ij}^*})^2 \prod_{i=1}^{\ell} \frac{1 - t^{2\xi_i}}{1 - t^{\xi_i}} = \prod_{(i,j) \in \xi^*} (1 - t^{h_{ij}^*})^2 \prod_{i=1}^{\ell} (1 + t^{\xi_i}).$$

Now the theorem follows from (5.22), (5.23), (5.24), and (5.25). \square

Remark 5.10. The formulas in Theorem 5.8 appear to differ by a factor $2^{\delta(\xi)}$ from [WW1, Theorem A] because of a different formulation due to the type \mathbf{Q} phenomenon.

5.5. The graded multiplicity in $\mathcal{C}l_n \otimes S^*V \otimes \wedge^*V$ and $\mathcal{C}l_n \otimes S^*V \otimes S^*V$. Similarly, we can consider the multiplicity of D^ξ for $\xi \in \mathcal{SP}_n$ in the bi-graded \mathcal{H}_n -modules $\mathcal{C}l_n \otimes S^*V \otimes \wedge^*V$ and $\mathcal{C}l_n \otimes S^*V \otimes S^*V$, and let

$$\begin{aligned}\widehat{d}_\xi(t, s) &= \dim \operatorname{Hom}_{\mathcal{H}_n}(D^\xi, \mathcal{C}l_n \otimes S_t V \otimes \wedge_s V), \\ \widetilde{d}_\xi(t, s) &= \dim \operatorname{Hom}_{\mathcal{H}_n}(D^\xi, \mathcal{C}l_n \otimes S_t V \otimes S_s V).\end{aligned}$$

Theorem 5.11. *Suppose $\xi \in \mathcal{SP}_n$. Then*

- (1) $\widehat{d}_\xi(t, s) = 2^{-\frac{\ell(\lambda) - \delta(\lambda)}{2}} Q_\xi(t^\bullet, st^\bullet).$
- (2) $\widetilde{d}_\xi(t, s) = 2^{-\frac{\ell(\lambda) - \delta(\lambda)}{2}} Q_\xi(t^\bullet s^\bullet).$

Part (1) here is [WW1, Theorem C] with a different proof, while (2) is new.

Proof. By Lemma 3.1 and the computation at the beginning of the proof of Theorem 5.4, we have

$$\operatorname{tr} \sigma|_{\mathcal{C}l_n \otimes S_t V \otimes \wedge_s V} = 2^{\ell(\mu)} \cdot \frac{(1 - (-s)^{\mu_1})(1 - (-s)^{\mu_2}) \dots (1 - (-s)^{\mu_\ell})}{(1 - t^{\mu_1})(1 - t^{\mu_2}) \dots (1 - t^{\mu_\ell})},$$

for any $\sigma \in \mathfrak{S}_n$ of cycle type $\mu = (\mu_1, \mu_2, \dots) \in \mathcal{OP}_n$. Applying the characteristic map $\operatorname{ch}^- : R^- \rightarrow \Gamma_{\mathbb{Q}}$, we obtain that

$$\begin{aligned}(5.26) \quad \operatorname{ch}^-(\mathcal{C}l_n \otimes S_t V \otimes \wedge_s V) &= \sum_{\mu \in \mathcal{OP}_n} z_\mu^{-1} \frac{2^{\ell(\mu)} (1 - (-s)^{\mu_1})(1 - (-s)^{\mu_2}) \dots (1 - (-s)^{\mu_\ell})}{(1 - t^{\mu_1})(1 - t^{\mu_2}) \dots (1 - t^{\mu_\ell})} p_\mu \\ &= [u^n] \sum_{\mu \in \mathcal{OP}} 2^{\ell(\mu)} z_\mu^{-1} u^{|\mu|} p_\mu(t^\bullet; st^\bullet) p_\mu \\ &= [u^n] \prod_{j \geq 0} \prod_i \frac{1 + ut^j z_i}{1 - ut^j z_i} \frac{1 + ust^j z_i}{1 - ust^j z_i} \\ &= \sum_{\lambda \in \mathcal{SP}_n} 2^{-\ell(\xi)} Q_\xi(t^\bullet, st^\bullet) Q_\xi(z),\end{aligned}$$

where the last two equalities used (3.10) and Cauchy identity from Theorem 3.4. It follows by (3.14) and the definition of $\widehat{d}_\xi(t, s)$ that

$$\operatorname{ch}^-(\mathcal{C}l_n \otimes S_t V \otimes \wedge_s V) = \sum_{\xi \in \mathcal{SP}_n} 2^{-\frac{\ell(\xi) + \delta(\xi)}{2}} \widehat{d}_\xi(t, s) Q_\xi(z).$$

Comparing these two different expressions for $\operatorname{ch}^-(\mathcal{C}l_n \otimes S_t V \otimes \wedge_s V)$ and noting the linear independence of the $Q_\xi(z)$'s, we prove (1).

Using a similar argument, one can verify (2) with the calculation of the character values of $S^*V \otimes S^*V$ in the proof of Theorem 5.5 at hand. \square

Remark 5.12. It will be interesting to find closed formulas for $s_\lambda(t^\bullet s^\bullet)$ and $Q_\xi(t^\bullet, st^\bullet)$.

6. SPIN KOSTKA POLYNOMIALS

In this section, following our very recent work [WW2] we introduce the spin Kostka polynomials, and show that the spin Kostka polynomials enjoy favorable properties parallel to the Kostka polynomials. Two interpretations of the spin Kostka polynomials in terms of graded multiplicities in the representation theory of Hecke-Clifford algebras and q -weight multiplicity for the queer Lie superalgebras are presented.

6.1. The ubiquitous Kostka polynomials. For $\lambda, \mu \in \mathcal{P}$, let $K_{\lambda\mu}$ be the Kostka number which counts the number of semistandard tableaux of shape λ and weight μ . We write $|\lambda| = n$ for $\lambda \in \mathcal{P}_n$. The dominance order on \mathcal{P} is defined by letting

$$\lambda \geq \mu \Leftrightarrow |\lambda| = |\mu| \text{ and } \lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i, \forall i \geq 1.$$

Let $\lambda, \mu \in \mathcal{P}$. The Kostka(-Foulkes) polynomial $K_{\lambda\mu}(t)$ is defined by

$$(6.1) \quad s_\lambda(x) = \sum_{\mu} K_{\lambda\mu}(t) P_\mu(x; t),$$

where $P_\mu(x; t)$ are the Hall-Littlewood functions (cf. [Mac, III, §2]). The following is a summary of some main properties of the Kostka polynomials.

Theorem 6.1. (cf. [Mac, III, §6]) *Suppose $\lambda, \mu \in \mathcal{P}_n$. Then the Kostka polynomials $K_{\lambda\mu}(t)$ satisfy the following properties:*

- (1) $K_{\lambda\mu}(t) = 0$ unless $\lambda \geq \mu$; $K_{\lambda\lambda}(t) = 1$.
- (2) The degree of $K_{\lambda\mu}(t)$ is $n(\mu) - n(\lambda)$.
- (3) $K_{\lambda\mu}(t)$ is a polynomial with non-negative integer coefficients.
- (4) $K_{\lambda\mu}(1) = K_{\lambda\mu}$.
- (5) $K_{(n)\mu}(t) = t^{n(\mu)}$.
- (6) $K_{\lambda(1^n)} = \frac{t^{n(\lambda')}(1-t)(1-t^2)\dots(1-t^n)}{\prod_{(i,j) \in \lambda} (1-t^{h_{ij}})}.$

Let \mathcal{B} be the flag variety for the general linear group $GL_n(\mathbb{C})$. For a partition μ of n , the Springer fiber \mathcal{B}_μ is the subvariety of \mathcal{B} consisting of flags preserved by the Jordan canonical form J_μ of shape μ . According to the Springer theory, the cohomology group $H^\bullet(\mathcal{B}_\mu)$ of \mathcal{B}_μ with complex coefficient affords a graded representation of \mathfrak{S}_n (which is the Weyl group of $GL_n(\mathbb{C})$). Define $C_{\lambda\mu}(t)$ to be the graded multiplicity

$$(6.2) \quad C_{\lambda\mu}(t) = \sum_{i \geq 0} t^i \operatorname{Hom}_{\mathfrak{S}_n}(S^\lambda, H^{2i}(\mathcal{B}_\mu)).$$

Theorem 6.2. (cf. [Mac, III, 7, Ex. 8], [GP, (5.7)]) *The following holds for $\lambda, \mu \in \mathcal{P}$:*

$$K_{\lambda\mu}(t) = C_{\lambda\mu}(t^{-1})t^{n(\mu)}.$$

Denote by $\{\epsilon_1, \dots, \epsilon_n\}$ the basis dual to the standard basis $\{E_{ii} \mid 1 \leq i \leq n\}$ in the standard Cartan subalgebra of $\mathfrak{gl}(n)$. For $\lambda, \mu \in \mathcal{P}$ with $\ell(\lambda) \leq n$ and $\ell(\mu) \leq n$, define the q -weight multiplicity of weight μ in an irreducible $\mathfrak{gl}(n)$ -module $L(\lambda)$ to be

$$m_\mu^\lambda(t) = [e^\mu] \frac{\prod_{\alpha > 0} (1 - e^{-\alpha})}{\prod_{\alpha > 0} (1 - te^{-\alpha})} \operatorname{ch} L(\lambda),$$

where the product $\prod_{\alpha > 0}$ is over all positive roots $\{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\}$ for $\mathfrak{gl}(n)$ and $[e^\mu]f(e^{\epsilon_1}, \dots, e^{\epsilon_n})$ denotes the coefficient of the monomial e^μ in a formal series $f(e^{\epsilon_1}, \dots, e^{\epsilon_n})$. A conjecture of Lusztig proved by Sato [Ka, Lu2] states that

$$(6.3) \quad K_{\lambda\mu}(t) = m_\mu^\lambda(t).$$

Let e be a regular nilpotent element in the Lie algebra $\mathfrak{gl}(n)$. For each $\mu \in \mathcal{P}$ with $\ell(\mu) \leq n$, define the Brylinski-Kostant filtration $\{J_e^k(L(\lambda)_\mu)\}_{k \geq 0}$ on the μ -weight space $L(\lambda)_\mu$ with

$$J_e^k(L(\lambda)_\mu) = \{v \in L(\lambda)_\mu \mid e^{k+1}v = 0\}.$$

Define a polynomial $\gamma_{\lambda\mu}(t)$ by letting

$$\gamma_{\lambda\mu}(t) = \sum_{k \geq 0} \left(\dim J_e^k(L(\lambda)_\mu) / J_e^{k-1}(L(\lambda)_\mu) \right) t^k.$$

The following theorem is due to R. Brylinski (see [Br, Theorem 3.4] and (6.3)).

Theorem 6.3. *Suppose $\lambda, \mu \in \mathcal{P}$ with $\ell(\lambda) \leq n$ and $\ell(\mu) \leq n$. Then we have*

$$K_{\lambda\mu}(t) = \gamma_{\lambda\mu}(t).$$

6.2. The spin Kostka polynomials. Denote by \mathbf{P}' the ordered alphabet $\{1' < 1 < 2' < 2 < 3' < 3 \cdots\}$. The symbols $1', 2', 3', \dots$ are said to be marked, and we shall denote by $|a|$ the unmarked version of any $a \in \mathbf{P}'$; that is, $|k'| = |k| = k$ for each $k \in \mathbb{N}$. For a strict partition ξ , a *marked shifted tableau* T of shape ξ , or a *marked shifted ξ -tableau* T , is an assignment $T : \xi^* \rightarrow \mathbf{P}'$ satisfying:

- (M1) The letters are weakly increasing along each row and column.
- (M2) The letters $\{1, 2, 3, \dots\}$ are strictly increasing along each column.
- (M3) The letters $\{1', 2', 3', \dots\}$ are strictly increasing along each row.

For a marked shifted tableau T of shape ξ , let α_k be the number of cells $(i, j) \in \xi^*$ such that $|T(i, j)| = k$ for $k \geq 1$. The sequence $(\alpha_1, \alpha_2, \alpha_3, \dots)$ is called the *weight* of T . The Schur Q -function associated to ξ can be interpreted as (see [Sag, St, Mac])

$$Q_\xi(x) = \sum_T x^T,$$

where the summation is taken over all marked shifted tableaux of shape ξ , and $x^T = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ if T has weight $(\alpha_1, \alpha_2, \alpha_3, \dots)$. Set

$$K_{\xi\mu}^- = \#\{T \mid T \text{ is a marked shifted tableau of shape } \xi \text{ and weight } \mu\}.$$

Then we have

$$(6.4) \quad Q_\xi(x) = \sum_\mu K_{\xi\mu}^- m_\mu(x),$$

where $K_{\xi\mu}^-$ is related to $\widehat{K}_{\xi\mu}$ appearing in Theorem 3.3 by

$$K_{\xi\mu}^- = 2^{\ell(\xi)} \widehat{K}_{\xi\mu}.$$

Definition 6.4. [WW2] The *spin Kostka polynomials* $K_{\xi\mu}^-(t)$ for $\xi \in \mathcal{SP}$ and $\mu \in \mathcal{P}$ are given by

$$(6.5) \quad Q_\xi(x) = \sum_{\mu} K_{\xi\mu}^-(t) P_\mu(x; t).$$

For $\xi \in \mathcal{SP}$, write

$$(6.6) \quad Q_\xi(x) = \sum_{\lambda \in \mathcal{P}} b_{\xi\lambda} s_\lambda(x),$$

for some suitable structure constants $b_{\xi\lambda}$.

Proposition 6.5. *The following holds for $\xi \in \mathcal{SP}$ and $\mu \in \mathcal{P}$:*

$$K_{\xi\mu}^-(t) = \sum_{\lambda \in \mathcal{P}} b_{\xi\lambda} K_{\lambda\mu}(t).$$

Proof. By (6.1) and (6.6), one can deduce that

$$\sum_{\mu} K_{\xi\mu}^-(t) P_\mu(x; t) = \sum_{\lambda, \mu \in \mathcal{P}_n} b_{\xi\lambda} K_{\lambda\mu}(t) P_\mu(x; t).$$

The proposition now follows from the fact that the Hall-Littlewood functions $P_\mu(x; t)$ are linearly independent in $\mathbb{Z}[t] \otimes_{\mathbb{Z}} \Lambda$. \square

The usual Kostka polynomial satisfies that $K_{\lambda\mu}(0) = \delta_{\lambda\mu}$. It follows from Proposition 6.5 that

$$K_{\xi\mu}^-(0) = b_{\xi\mu}.$$

For $\xi \in \mathcal{SP}, \lambda \in \mathcal{P}$, set

$$(6.7) \quad g_{\xi\lambda} = 2^{-\ell(\xi)} b_{\xi\lambda}.$$

Up to some 2-power, $g_{\xi\lambda}$ has the following interpretation of branching coefficient for the restriction of a $\mathfrak{gl}(n)$ -module $V(\lambda)$ to $\mathfrak{gl}(n)$.

Lemma 6.6. *As a $\mathfrak{gl}(n)$ -module, $V(\xi)$ can be decomposed as*

$$V(\xi) \cong \bigoplus_{\lambda \in \mathcal{P}, \ell(\lambda) \leq n} 2^{\frac{\ell(\xi) + \delta(\xi)}{2}} g_{\xi\lambda} L(\lambda).$$

Proof. It suffices to verify on the character level. The corresponding character identity indeed follows from (6.6), (6.7) and Theorem 4.9, as the character of $L(\lambda)$ is given by the Schur function s_λ . \square

Lemma 6.7. [St, Theorem 9.3] [Mac, III, (8.17)] *The following holds for $\xi \in \mathcal{SP}, \lambda \in \mathcal{P}$:*

$$(6.8) \quad g_{\xi\lambda} \in \mathbb{Z}_+; \quad g_{\xi\lambda} = 0 \text{ unless } \xi \geq \lambda; \quad g_{\xi\xi} = 1.$$

Stembridge [St] proved Lemma 6.7 by providing a combinatorial formula for $g_{\xi\lambda}$ in terms of marked shifted tableaux. We give a representation theoretic proof below.

Proof. It follows by Lemma 6.6 that $g_{\xi\lambda} \geq 0$, and moreover, $g_{\xi\lambda} = 0$ unless $\xi \geq \lambda$ (the dominance order for compositions coincide with the dominance order of weights for $\mathfrak{q}(n)$). The highest weight space for the $\mathfrak{q}(n)$ -module $V(\xi)$ is W_ξ , which has dimension $2^{\frac{\ell(\xi)+\delta(\xi)}{2}}$. Hence, $g_{\xi\xi} = 1$, by Lemma 6.6 again.

By Theorem 4.9, $2^{-\frac{\ell(\xi)+\delta(\xi)}{2}} \text{ch} V(\xi) = 2^{-\ell(\xi)} Q(x_1, \dots, x_n)$, which is known to lie in Λ , cf. [Mac] (this fact can also be seen directly from representation theory of $\mathfrak{q}(n)$). Hence, $2^{-\ell(\xi)} Q(x_1, \dots, x_n)$ is a \mathbb{Z} -linear combination of Schur polynomials s_λ . Combining with Lemma 6.6, this proves that $g_{\xi\lambda} \in \mathbb{Z}$. \square

The following is a spin counterpart of the properties of Kostka polynomials listed in Theorem 6.1.

Theorem 6.8. [WW2] *The spin Kostka polynomials $K_{\xi\mu}^-(t)$ for $\xi \in \mathcal{SP}_n, \mu \in \mathcal{P}_n$ satisfy the following properties:*

- (1) $K_{\xi\mu}^-(t) = 0$ unless $\xi \geq \mu$; $K_{\xi\xi}^-(t) = 2^{\ell(\xi)}$.
- (2) The degree of the polynomial $K_{\xi\mu}^-(t)$ is $n(\mu) - n(\xi)$.
- (3) $2^{-\ell(\xi)} K_{\xi\mu}^-(t)$ is a polynomial with non-negative integer coefficients.
- (4) $K_{\xi\mu}^-(1) = K_{\xi\mu}^-$; $K_{\xi\mu}^-(-1) = 2^{\ell(\xi)} \delta_{\xi\mu}$.
- (5) $K_{(n)\mu}^-(t) = t^{n(\mu)} \prod_{i=1}^{\ell(\mu)} (1 + t^{1-i})$.
- (6) $K_{\xi(1^n)}^-(t) = \frac{t^{n(\xi)}(1-t)(1-t^2) \cdots (1-t^n) \prod_{(i,j) \in \xi^*} (1 + t^{c_{ij}})}{\prod_{(i,j) \in \xi^*} (1 - t^{h_{ij}^*})}$.

Proof. Combining Theorem 6.1(1)-(3), Lemma 6.7 and Proposition 6.5, we can easily verify that the spin Kostka polynomial $K_{\xi\mu}^-(t)$ must satisfy the properties (1)-(3) in the theorem. It is known that $P_\mu(x; 1) = m_\mu$ and hence by (6.4) we have $K_{\xi\mu}^-(1) = K_{\xi\mu}^-$. Also, $Q_\xi = 2^{\ell(\xi)} P_\xi(x; -1)$, and $\{P_\mu(x; -1) \mid \mu \in \mathcal{P}\}$ forms a basis for Λ (see [Mac, p.253]). Hence (4) is proved.

By [Mac, III, §3, Example 1(3)] we have

$$(6.9) \quad \prod_{i \geq 1} \frac{1 + x_i}{1 - x_i} = \sum_{\mu} t^{n(\mu)} \prod_{j=1}^{\ell(\mu)} (1 + t^{1-j}) P_\mu(x; t).$$

Comparing the degree n terms of (6.9) and (3.6), we obtain that

$$Q_{(n)}(x) = q_n(x) = \sum_{\mu \in \mathcal{P}_n} t^{n(\mu)} \prod_{j=1}^{\ell(\mu)} (1 + t^{1-j}) P_\mu(x; t).$$

Hence (5) is proved.

Part (6) actually follows from Theorem 5.8 and Theorem 6.10 in Section 6.3 below, and let us postpone its proof after completing the proof of Theorem 6.10. \square

6.3. Spin Kostka polynomials and graded multiplicity. Recall the characteristic map ch and ch^- from (3.1) and (3.12). Note that ch^- is related to ch as follows:

$$(6.10) \quad \text{ch}^-(\zeta) = \text{ch}(\text{res}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} \zeta), \quad \text{for } \zeta \in R_n^-.$$

Recall that the \mathfrak{S}_n -module S^λ and \mathcal{H}_n -module D^ξ have characters given by χ^λ and ζ^ξ , respectively. Up to some 2-power as in Lemma 6.6, $g_{\xi\lambda}$ has another representation theoretic interpretation.

Lemma 6.9. *Suppose $\xi \in \mathcal{SP}_n, \lambda \in \mathcal{P}_n$. The following holds:*

$$\dim \operatorname{Hom}_{\mathcal{H}_n}(D^\xi, \operatorname{ind}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} S^\lambda) = 2^{\frac{\ell(\xi) + \delta(\xi)}{2}} g_{\xi\lambda}.$$

Proof. Since the \mathcal{H}_n -module $\operatorname{ind}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} S^\lambda$ is semisimple, we have

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{H}_n}(D^\xi, \operatorname{ind}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} S^\lambda) &= \dim \operatorname{Hom}_{\mathcal{H}_n}(\operatorname{ind}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} S^\lambda, D^\xi) \\ &= \dim \operatorname{Hom}_{\mathbb{C}\mathfrak{S}_n}(S^\lambda, \operatorname{res}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} D^\xi) \\ &= (s_\lambda, \operatorname{ch}(\operatorname{res}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} D^\xi)) \\ &= (s_\lambda, \operatorname{ch}^-(D^\xi)) \\ &= (s_\lambda, 2^{-\frac{\ell(\xi) - \delta(\xi)}{2}} Q_\xi(x)) \\ &= 2^{\frac{\ell(\xi) + \delta(\xi)}{2}} g_{\xi\lambda}, \end{aligned}$$

where the second equation uses the Frobenius reciprocity, the third equation uses the fact that ch is an isometry, the fourth, fifth and sixth equations follow from (6.10), (3.14) and (6.6), respectively. \square

For $\mu \in \mathcal{P}_n$ and $\xi \in \mathcal{SP}_n$, recalling (6.2), we define a polynomial $C_{\xi\mu}^-(t)$ as a graded multiplicity of the graded \mathcal{H}_n -module $\operatorname{ind}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} H^\bullet(\mathcal{B}_\mu) \cong \mathcal{C}l_n \otimes H^\bullet(\mathcal{B}_\mu)$:

$$(6.11) \quad C_{\xi\mu}^-(t) := \sum_{i \geq 0} t^i \left(\dim \operatorname{Hom}_{\mathcal{H}_n}(D^\xi, \mathcal{C}l_n \otimes H^{2i}(\mathcal{B}_\mu)) \right).$$

Theorem 6.10. [WW2] *Suppose $\xi \in \mathcal{SP}_n, \mu \in \mathcal{P}_n$. Then we have*

$$K_{\xi\mu}^-(t) = 2^{\frac{\ell(\xi) - \delta(\xi)}{2}} C_{\xi\mu}^-(t^{-1}) t^{n(\mu)}.$$

Proof. By Proposition 6.5 and Theorem 6.2, we obtain that

$$K_{\xi\mu}^-(t) = \sum_{\lambda \in \mathcal{P}_n} b_{\xi\lambda} K_{\lambda\mu}(t) = \sum_{\lambda \in \mathcal{P}_n} b_{\xi\lambda} C_{\lambda\mu}(t^{-1}) t^{n(\mu)}.$$

On the other hand, we have by Lemma 6.9 that

$$\begin{aligned} C_{\xi\mu}^-(t) &= \sum_{i \geq 0} t^i \left(\dim \operatorname{Hom}_{\mathcal{H}_n}(D^\xi, \operatorname{ind}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} H^{2i}(\mathcal{B}_\mu)) \right) \\ &= \sum_{\lambda} C_{\lambda\mu}(t) \dim \operatorname{Hom}_{\mathcal{H}_n}(D^\xi, \operatorname{ind}_{\mathbb{C}\mathfrak{S}_n}^{\mathcal{H}_n} S^\lambda) \\ &= 2^{-\frac{\ell(\xi) - \delta(\xi)}{2}} \sum_{\lambda \in \mathcal{P}_n} b_{\xi\lambda} C_{\lambda\mu}(t). \end{aligned}$$

Now the theorem follows by comparing the above two identities. \square

With Theorem 6.10 at hand, we can complete the proof of Theorem 6.8(6).

Proof of Theorem 6.8(6). Suppose $\xi \in \mathcal{SP}_n$. Observe that $\mathcal{B}_{(1^n)} = \mathcal{B}$ and it is well known that $H^\bullet(\mathcal{B})$ is isomorphic to the coinvariant algebra of the symmetric group \mathfrak{S}_n . Hence by Theorem 5.8 we obtain that

$$C_{\xi(1^n)}^-(t) = d^\xi(t) = 2^{-\frac{\ell(\xi) - \delta(\xi)}{2}} \frac{t^{n(\xi)}(1-t)(1-t^2) \cdots (1-t^n) \prod_{(i,j) \in \xi^*} (1+t^{c_{ij}})}{\prod_{(i,j) \in \xi^*} (1-t^{h_{ij}^*})},$$

where ξ^* is the shifted diagram associated to ξ , c_{ij}, h_{ij}^* are contents and shifted hook lengths for a cell $(i, j) \in \xi$. This together with Theorem 6.10 gives us

$$\begin{aligned} K_{\xi(1^n)}^-(t) &= \frac{t^{\frac{n(n-1)}{2} - n(\xi)}(1-t^{-1})(1-t^{-2}) \cdots (1-t^{-n}) \prod_{(i,j) \in \xi^*} (1+t^{-c_{ij}})}{\prod_{(i,j) \in \xi^*} (1-t^{-h_{ij}^*})} \\ &= \frac{t^{-n-n(\xi) + \sum_{(i,j) \in \xi^*} h_{ij}^*} (1-t)(1-t^2) \cdots (1-t^n) \prod_{(i,j) \in \xi^*} (1+t^{c_{ij}})}{t^{\sum_{(i,j) \in \xi^*} c_{ij}} \prod_{(i,j) \in \xi^*} (1-t^{h_{ij}^*})} \\ &= \frac{t^{n(\xi)}(1-t)(1-t^2) \cdots (1-t^n) \prod_{(i,j) \in \xi^*} (1+t^{c_{ij}})}{\prod_{(i,j) \in \xi^*} (1-t^{h_{ij}^*})}, \end{aligned}$$

where the last equality can be derived by noting that the contents c_{ij} are $0, 1, \dots, \xi_i - 1$ and the fact (cf. [Mac, III, §8, Example 12]) that in the i th row of ξ^* , the hook lengths h_{ij}^* for $i \leq j \leq \xi_i + i - 1$ are $1, 2, \dots, \xi_i, \xi_i + \xi_{i+1}, \xi_i + \xi_{i+2}, \dots, \xi_i + \xi_\ell$ with exception $\xi_i - \xi_{i+1}, \xi_i - \xi_{i+2}, \dots, \xi_i - \xi_\ell$. \square

6.4. Spin Kostka polynomials and q -weight multiplicity. Observe that there is a natural isomorphism $\mathfrak{q}(n)_{\bar{0}} \cong \mathfrak{gl}(n)$. Regarding a regular nilpotent element e in $\mathfrak{gl}(n)$ as an even element in $\mathfrak{q}(n)$, for $\xi \in \mathcal{SP}, \mu \in \mathcal{P}$ with $\ell(\xi) \leq n, \ell(\mu) \leq n$, we define a *Brylinski-Kostant filtration* $\{J_e^k(V(\xi)_\mu)\}_{k \geq 0}$ on the μ -weight space $V(\xi)_\mu$ of the irreducible $\mathfrak{q}(n)$ -module $V(\xi)$, where

$$J_e^k(V(\xi)_\mu) := \{v \in V(\xi)_\mu \mid e^{k+1}v = 0\}.$$

Define a polynomial $\gamma_{\xi\mu}^-(t)$ by letting

$$\gamma_{\xi\mu}^-(t) = \sum_{k \geq 0} \left(\dim J_e^k(V(\xi)_\mu) / J_e^{k-1}(V(\xi)_\mu) \right) t^k.$$

We are ready to establish the Lie theoretic interpretation of spin Kostka polynomials.

Theorem 6.11. [WW2] *Suppose $\xi \in \mathcal{SP}, \mu \in \mathcal{P}$ with $\ell(\xi) \leq n, \ell(\mu) \leq n$. Then we have*

$$K_{\xi\mu}^-(t) = 2^{\frac{\ell(\xi) - \delta(\xi)}{2}} \gamma_{\xi\mu}^-(t).$$

Proof. The Brylinski-Kostant filtration is defined via a regular nilpotent element in $\mathfrak{gl}(n) \cong \mathfrak{q}(n)_{\bar{0}}$, and thus it is compatible with the decomposition in Lemma 6.6. Hence, we have $J_e^k(V(\xi)_\mu) \cong \oplus_\lambda 2^{\frac{\ell(\xi) + \delta(\xi)}{2}} g_{\xi\lambda} J_e^k(L(\lambda)_\mu)$. It follows by the definitions of the

polynomials $\gamma_{\xi\mu}^-(t)$ and $\gamma_{\lambda\mu}(t)$ that

$$\gamma_{\xi\mu}^-(t) = \sum_{\lambda} 2^{\frac{\ell(\xi)+\delta(\xi)}{2}} g_{\xi\lambda} \gamma_{\lambda\mu}(t).$$

Then by Theorem 6.3 we obtain that

$$\gamma_{\xi\mu}^-(t) = \sum_{\lambda} 2^{\frac{\ell(\xi)+\delta(\xi)}{2}} g_{\xi\lambda} K_{\lambda\mu}(t) = \sum_{\lambda} 2^{-\frac{\ell(\xi)-\delta(\xi)}{2}} b_{\xi\lambda} K_{\lambda\mu}(t).$$

This together with Proposition 6.5 proves the theorem. \square

Remark 6.12. We can define spin Hall-Littlewood functions $H_{\mu}^-(x; t)$ via the spin Kostka polynomials as well as spin Macdonald polynomials $H_{\mu}^-(x; q, t)$ and the spin q, t -Kostka polynomials $K_{\xi\mu}^-(q, t)$. The use of Φ and φ makes such a q, t -generalization possible (see [WW2] for details). There is also a completely different vertex operator approach developed by Tudose and Zabrocki [TZ] toward a different version of spin Kostka polynomials and spin Hall-Littlewood functions, which did not seem to admit representation theoretic interpretation.

7. THE SEMINORMAL FORM CONSTRUCTION

In this section, we formulate the seminormal form for the irreducible \mathcal{H}_n -modules, analogous to Young's seminormal form for the irreducible $\mathbb{C}\mathfrak{S}_n$ -modules. Following the independent works of [HKS] and [Wan] (which was built on the earlier work of Nazarov [Naz]), we first work on the generality of affine Hecke-Clifford algebras, and then specialize to the (finite) Hecke-Clifford algebras to give an explicit construction of Young's seminormal form for the irreducible \mathcal{H}_n -modules.

7.1. Jucys-Murphy elements and Young's seminormal form for \mathfrak{S}_n . The Jucys-Murphy elements in the group algebra of the symmetric group \mathfrak{S}_n are defined by

$$(7.1) \quad L_k = \sum_{1 \leq j < k} (j, k),$$

where (j, k) is the transposition between j and k . Observe that L_k is the difference between the sum of all transpositions in \mathfrak{S}_k and the sum of all transpositions in \mathfrak{S}_{k-1} . Hence the Jucys-Murphy elements L_1, \dots, L_n commute and act semisimply on irreducible $\mathbb{C}\mathfrak{S}_n$ -modules.

The Gelfand-Zetlin subalgebra \mathcal{A}_n of $\mathbb{C}\mathfrak{S}_n$ is defined to be the subalgebra consisting of the diagonal matrices in the Wedderburn decomposition of $\mathbb{C}\mathfrak{S}_n$. It is not difficult to show by induction on n (see [OV, Corollary 4.1] and [Kle, Lemma 2.1.4]) that \mathcal{A}_n is generated by the centers of the subalgebras $\mathbb{C}\mathfrak{S}_1, \mathbb{C}\mathfrak{S}_2, \dots, \mathbb{C}\mathfrak{S}_n$, and that it is also generated by the Jucys-Murphy elements L_1, \dots, L_n .

The moral is that the subalgebra \mathcal{A}_n of $\mathbb{C}\mathfrak{S}_n$ plays a role of a Cartan subalgebra of a semisimple Lie algebra. Every irreducible $\mathbb{C}\mathfrak{S}_n$ -module V can be decomposed as

$$V = \bigoplus_{\underline{i}=(i_1, \dots, i_n) \in \mathbb{C}^n} V_{\underline{i}},$$

where $V_{\underline{i}} = \{v \in V \mid L_k v = i_k v, 1 \leq k \leq n\}$ is the simultaneous eigenspace of L_1, \dots, L_n with eigenvalues i_1, \dots, i_n . By the description of \mathcal{A}_n above, we have either $V_{\underline{i}} = 0$ or $\dim V_{\underline{i}} = 1$. If $V_{\underline{i}} \neq 0$, we say that \underline{i} is a weight of V and $V_{\underline{i}}$ is the \underline{i} -weight space of V , and we fix a nonzero vector $v_{\underline{i}} \in V_{\underline{i}}$.

Suppose $\lambda \in \mathcal{P}_n$ and T is a standard tableau of shape λ . Define its content sequence $c(T) = (c(T_1), \dots, c(T_n)) \in \mathbb{C}^n$ by letting $c(T_k)$ be the content of the cell occupied by k in T for $1 \leq k \leq n$. By analyzing the structures of weights, we can show that the sequences $c(T)$ for standard tableaux T with n cells are exactly all the weights for irreducible \mathfrak{S}_n -modules. Now we are ready to formulate the Young's seminormal form for irreducible $\mathbb{C}\mathfrak{S}_n$ -modules. For $\lambda \in \mathcal{P}_n$, define $V^\lambda = \sum_T \mathbb{C}v_T$, where the summation is taken over standard tableaux of shape λ . For $1 \leq k \leq n-1$, define

$$(7.2) \quad s_k v_T = (c(T_{k+1}) - c(T_k))^{-1} v_T + \sqrt{1 - (c(T_{k+1}) - c(T_k))^{-2}} v_{s_k T},$$

where $s_k T$ indicates the standard tableau obtained by switching k and $k+1$ in T and $v_{s_k T} = 0$ if $s_k T$ is not standard. In this way Okounkov and Vershik [OV] established the following.

Theorem 7.1 (Young's seminormal form). *For $\lambda \in \mathcal{P}_n$, V^λ affords an irreducible \mathfrak{S}_n -module given by (7.2). Moreover, $\{V^\lambda \mid \lambda \in \mathcal{P}_n\}$ forms a complete set of non-isomorphic irreducible \mathfrak{S}_n -modules.*

7.2. Jucys-Murphy elements for \mathcal{H}_n . As in the group algebra of symmetric groups, there also exist Jucys-Murphy elements $J_k (1 \leq k \leq n)$ in \mathcal{H}_n defined as (see [Naz])

$$(7.3) \quad J_k = \sum_{1 \leq j < k} (1 + c_j c_k)(j, k).$$

Lemma 7.2. *The following holds:*

- (1) $J_i J_k = J_k J_i$, for $1 \leq i \neq k \leq n$.
- (2) $c_i J_i = -J_i c_i$, $c_i J_k = J_k c_i$, for $1 \leq i \neq k \leq n$.
- (3) $s_i J_i = J_{i+1} s_i - (1 + c_i c_{i+1})$, for $1 \leq i \leq n-1$.
- (4) $s_i J_k = J_k s_i$, for $k \neq i, i+1$.

Proof. It follows by a direct computation that $c_i J_n = J_n c_i$, and $\sigma J_n = J_n \sigma$, for $1 \leq i \leq n-1$ and $\sigma \in \mathfrak{S}_{n-1}$. Hence, J_n commutes with \mathcal{H}_{n-1} , and whence (1). The remaining properties can be also verified by direct calculations. \square

7.3. Degenerate affine Hecke-Clifford algebras $\mathcal{H}_n^{\text{aff}}$. For $n \in \mathbb{Z}_+$, the affine Hecke-Clifford algebra $\mathcal{H}_n^{\text{aff}}$ is defined to be the superalgebra generated by even generators $s_1, \dots, s_{n-1}, x_1, \dots, x_n$ and odd generators c_1, \dots, c_n subject to the following relations (besides the relations (2.1), (2.3) and (2.4)):

$$(7.4) \quad x_i x_j = x_j x_i, \quad 1 \leq i, j \leq n,$$

$$(7.5) \quad s_i x_i = x_{i+1} s_i - (1 + c_i c_{i+1}), \quad 1 \leq i \leq n-1,$$

$$(7.6) \quad s_i x_j = x_j s_i, \quad j \neq i, i+1, \quad 1 \leq i, j \leq n,$$

$$(7.7) \quad x_i c_i = -c_i x_i, \quad x_i c_j = c_j x_i, \quad 1 \leq i \neq j \leq n.$$

Remark 7.3. The affine Hecke-Clifford algebra $\mathcal{H}_n^{\text{aff}}$ was introduced by Nazarov [Naz] (sometimes called affine Sergeev algebra). The Morita super-equivalence (2.5) between \mathcal{H}_n and $\mathbb{C}\mathfrak{S}_n^-$ has been extended to one between $\mathcal{H}_n^{\text{aff}}$ and the affine spin Hecke algebras [Wa1, Proposition 3.4] and for other classical type Weyl groups [KW, Theorem 4.4].

Denote by \mathcal{P}_n^c the superalgebra generated by even generators x_1, \dots, x_n and odd generators c_1, \dots, c_n subject to the relations (2.3), (7.4) and (7.7). For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ and $\beta \in \mathbb{Z}_2^n$, set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $c^\beta = c_1^{\beta_1} \cdots c_n^{\beta_n}$. Then we have the following.

Lemma 7.4. [Naz] [BK, Theorem 2.2] *The set $\{x^\alpha c^\beta w \mid \alpha \in \mathbb{Z}_+^n, \beta \in \mathbb{Z}_2^n, w \in \mathfrak{S}_n\}$ forms a basis of $\mathcal{H}_n^{\text{aff}}$.*

Sketch of a proof. One can construct a representation π of $\mathcal{H}_n^{\text{aff}}$ on the polynomial-Clifford algebra \mathcal{P}_n^c , where the x_i and c_i for all i act by left multiplication (and the action of s_i 's is then determined uniquely). Then one checks that the linear operators $\pi(x^\alpha c^\beta w)$ are linearly independent. We refer to the proof of [KW, Theorem 3.4] for detail. \square

By [Naz], there exists a surjective homomorphism

$$(7.8) \quad F : \mathcal{H}_n^{\text{aff}} \longrightarrow \mathcal{H}_n$$

$$c_k \mapsto c_k, s_l \mapsto s_l, x_k \mapsto J_k, \quad (1 \leq k \leq n, 1 \leq l \leq n-1),$$

and the kernel of F coincides with the ideal $\langle x_1 \rangle$ of $\mathcal{H}_n^{\text{aff}}$ generated by x_1 . Hence the category of finite-dimensional \mathcal{H}_n -modules can be identified as the category of finite-dimensional $\mathcal{H}_n^{\text{aff}}$ -modules which are annihilated by x_1 . For the study of $\mathcal{H}_n^{\text{aff}}$ -modules, we shall mainly focus on the so-called finite-dimensional *integral* modules, on which x_1^2, \dots, x_n^2 have eigenvalues of the form

$$q(i) = i(i+1), \quad i \in \mathbb{Z}_+.$$

It is easy to see that a finite-dimensional $\mathcal{H}_n^{\text{aff}}$ -module M is integral if all of eigenvalues of x_j^2 for a fixed j on M are of the form $q(i)$ (cf. [BK, Lemma 4.4] or [Kle, Lemma 15.1.2]). Hence the category of finite-dimensional \mathcal{H}_n -modules can be identified with the subcategory of integral $\mathcal{H}_n^{\text{aff}}$ -modules on which $x_1 = 0$.

By Lemma 7.4, \mathcal{P}_n^c can be identified with the subalgebra of $\mathcal{H}_n^{\text{aff}}$ generated by x_1, \dots, x_n and c_1, \dots, c_n . For $i \in \mathbb{Z}_+$, denote by $L(i)$ the 2-dimensional \mathcal{P}_1^c -module with $L(i)_{\bar{0}} = \mathbb{C}v_0$ and $L(i)_{\bar{1}} = \mathbb{C}v_1$ and

$$x_1 v_0 = \sqrt{q(i)} v_0, \quad x_1 v_1 = -\sqrt{q(i)} v_1, \quad c_1 v_0 = v_1, \quad c_1 v_1 = v_0.$$

Note that $L(i)$ is irreducible of type \mathbf{M} if $i \neq 0$, and irreducible of type \mathbf{Q} if $i = 0$. Moreover $L(i), i \in \mathbb{Z}_+$ form a complete set of pairwise non-isomorphic integral irreducible \mathcal{P}_1^c -module. Since $\mathcal{P}_n^c \cong \mathcal{P}_1^c \otimes \cdots \otimes \mathcal{P}_1^c$, Lemma 2.5 implies the following.

Lemma 7.5. $\{L(\underline{i}) = L(i_1) \otimes L(i_2) \otimes \cdots \otimes L(i_n) \mid \underline{i} = (i_1, \dots, i_n) \in \mathbb{Z}_+^n\}$ form a complete set of pairwise non-isomorphic integral irreducible \mathcal{P}_n^c -modules. Furthermore, $\dim L(\underline{i}) = 2^{n - \lfloor \frac{\gamma_0}{2} \rfloor}$, where γ_0 denotes the number of $1 \leq j \leq n$ with $i_j = 0$, and $\lfloor \frac{\gamma_0}{2} \rfloor$ denotes the greatest integer less than or equal to $\frac{\gamma_0}{2}$.

The following definition of [HKS, Wan] is motivated by similar studies for the affine Hecke algebras in [Ch, Ram, Ru].

Definition 7.6. A representation of $\mathcal{H}_n^{\text{aff}}$ is called *completely splittable* if x_1, \dots, x_n act semisimply.

Since the polynomial generators x_1, \dots, x_n commute, a finite-dimensional integral completely splittable $\mathcal{H}_n^{\text{aff}}$ -module M can be decomposed as

$$M = \bigoplus_{\underline{i} \in \mathbb{Z}_+^n} M_{\underline{i}},$$

where

$$M_{\underline{i}} = \{z \in M \mid x_k^2 z = q(i_k)z, 1 \leq k \leq n\}.$$

If $M_{\underline{i}} \neq 0$, then \underline{i} is called a *weight* of M and $M_{\underline{i}}$ is called a *weight space*. Since $x_k^2, 1 \leq k \leq n$ commute with c_1, \dots, c_n , each $M_{\underline{i}}$ is actually \mathcal{P}_n^c -submodule of M .

Following Nazarov, we define the intertwining elements as

$$(7.9) \quad \phi_k := s_k(x_k^2 - x_{k+1}^2) + (x_k + x_{k+1}) + c_k c_{k+1}(x_k - x_{k+1}), 1 \leq k < n.$$

It is known [Naz] and easy to check directly that

$$(7.10) \quad \phi_k^2 = 2(x_k^2 + x_{k+1}^2) - (x_k^2 - x_{k+1}^2)^2,$$

$$(7.11) \quad \phi_k x_k = x_{k+1} \phi_k, \phi_k x_{k+1} = x_k \phi_k, \phi_k x_l = x_l \phi_k,$$

$$(7.12) \quad \phi_k c_k = c_{k+1} \phi_k, \phi_k c_{k+1} = c_k \phi_k, \phi_k c_l = c_l \phi_k,$$

$$(7.13) \quad \phi_j \phi_k = \phi_k \phi_j, \phi_k \phi_{k+1} \phi_k = \phi_{k+1} \phi_k \phi_{k+1},$$

for all admissible j, k, l with $l \neq k, k+1$ and $|j - k| > 1$.

7.4. Weights and standard skew shifted tableaux. This subsection is technical though elementary in nature, and we recommend the reader to skip most of the proofs in a first reading. The upshot of this subsection is Proposition 7.12 which identifies the weights as content vectors associated to standard skew shifted tableaux.

Lemma 7.7. Suppose that M is an integral completely splittable $\mathcal{H}_n^{\text{aff}}$ -module and that $\underline{i} = (i_1, \dots, i_n) \in \mathbb{Z}_+^n$ is a weight of M . Then $i_k \neq i_{k+1}$ for all $1 \leq k \leq n-1$.

Proof. Suppose $i_k = i_{k+1}$ for some $1 \leq k \leq n-1$. Let $0 \neq z \in M_{\underline{i}}$. One can show using (7.5) that

$$(7.14) \quad x_k^2 s_k = s_k x_{k+1}^2 - (x_k(1 - c_k c_{k+1}) + (1 - c_k c_{k+1})x_{k+1})$$

$$(7.15) \quad x_{k+1}^2 s_k = s_k x_k^2 + (x_{k+1}(1 + c_k c_{k+1}) + (1 + c_k c_{k+1})x_k).$$

Since M is completely splittable, $(x_k^2 - q(i_k))z = 0 = (x_{k+1}^2 - q(i_{k+1}))z$. This together with (7.14) shows that

$$(7.16) \quad (x_k^2 - q(i_k))s_k z = (x_k^2 - q(i_{k+1}))s_k z = -(x_k(1 - c_k c_{k+1}) + (1 - c_k c_{k+1})x_{k+1})z,$$

and hence

$$(x_k^2 - q(i_k))^2 s_k z = -(x_k(1 - c_k c_{k+1}) + (1 - c_k c_{k+1})x_{k+1})(x_k^2 - q(i_k))z = 0.$$

Similarly, we see that

$$(x_{k+1}^2 - q(i_{k+1}))^2 s_k z = 0.$$

Hence $s_k z \in M_{\underline{i}}$, i.e., $(x_k^2 - q(i_k))s_k z = 0$, and therefore (7.16) implies that

$$\begin{aligned} (x_k(1 - c_k c_{k+1}) + (1 - c_k c_{k+1})x_{k+1})z &= 0, \\ 2(x_k^2 + x_{k+1}^2)z &= (x_k(1 - c_k c_{k+1}) + (1 - c_k c_{k+1})x_{k+1})^2 z = 0. \end{aligned}$$

This means that $q(i_{k+1}) = -q(i_k)$ and hence $q(i_k) = q(i_{k+1}) = 0$ since $i_k = i_{k+1}$. We conclude that $x_k = 0 = x_{k+1}$ on $M_{\underline{i}}$. This implies that $x_{k+1}s_k z = 0$ since $s_k z \in M_{\underline{i}}$ as shown above. Then

$$(1 + c_k c_{k+1})z = x_{k+1}s_k z - s_k x_k z = 0,$$

and hence $z = \frac{1}{2}(1 - c_k c_{k+1})(1 + c_k c_{k+1})z = 0$, which is a contradiction. \square

Lemma 7.8. *Assume that $\underline{i} = (i_1, \dots, i_n) \in \mathbb{Z}_+^n$ is a weight of an irreducible integral completely splittable $\mathcal{H}_n^{\text{aff}}$ -module M . Fix $1 \leq k \leq n-1$.*

- (1) *If $i_k \neq i_{k+1} \pm 1$, then $\phi_k z$ is a nonzero weight vector of weight $s_k \cdot \underline{i}$ for any $0 \neq z \in M_{\underline{i}}$. Hence $s_k \cdot \underline{i}$ is a weight of M .*
- (2) *If $i_k = i_{k+1} \pm 1$, then $\phi_k = 0$ on $M_{\underline{i}}$.*

Proof. It follows from (7.11) that $\phi_k M_{\underline{i}} \subseteq M_{s_k \cdot \underline{i}}$. By (7.10), we have

$$\phi_k^2 z = (2(x_k^2 + x_{k+1}^2) - (x_k^2 - x_{k+1}^2)^2)z = (2(q(i_k) + q(i_{k+1})) - (q(i_k) - q(i_{k+1}))^2)z$$

for any $z \in M_{\underline{i}}$. A calculation shows that $2(q(i_k) + q(i_{k+1})) - (q(i_k) - q(i_{k+1}))^2 \neq 0$ when $i_k \neq i_{k+1} \pm 1$ and hence $\phi_k^2 z \neq 0$. This proves (1).

Assume now that $i_k = i_{k+1} \pm 1$. Suppose $\phi_k z \neq 0$ for some $z \in M_{\underline{i}}$. Since M is irreducible, there exists a sequence $1 \leq a_1, a_2, \dots, a_m \leq n-1$ such that

$$(7.17) \quad \phi_{a_m} \cdots \phi_{a_2} \phi_{a_1} \phi_k z = \alpha z$$

for some $0 \neq \alpha \in \mathbb{C}$. Assume that m is minimal such that (7.17) holds. Let $\sigma = s_{a_m} \cdots s_{a_1} s_k \in \mathfrak{S}_n$. Then $\sigma \cdot \underline{i} = \underline{i}$. If $\sigma \neq 1$, then there exists $1 \leq b_1 \leq b_2 \leq n$ such that $i_{b_1} = i_{b_2}$, and $\sigma = (i_1, i_2)$ by the minimality of m . Hence, i_{b_1} and i_{b_2} can be brought to be adjacent by the permutation $s_{a_j} \cdots s_{a_1} s_k \cdot \underline{i}$ for some $1 \leq j \leq m$. That is, $s_{a_j} \cdots s_{a_1} s_k \cdot \underline{i}$ is a weight of M of the form $(\dots, \beta, \beta, \dots)$, which contradicts Lemma 7.7. Hence $\sigma = 1$ and $s_{a_m} \cdots s_{a_2} s_{a_1} = s_k$. We further claim that $m = 1$. Suppose that $m > 1$. Then, by the exchange condition for Coxeter groups, there exists $1 \leq p < q \leq m$ such that $s_{a_m} \cdots s_{a_q} \cdots s_{a_p} \cdots s_{a_1} = s_{a_m} \cdots \check{s}_{a_q} \cdots \check{s}_{a_p} \cdots s_{a_1}$, where \check{s} means the very term is removed. This leads to an identity similar to (7.17) for a product of $(m-1)$ ϕ 's, contradicting the minimality of m . Therefore $m = 1$ and then $a_1 = k$, which together with (7.17) leads to $\phi_k^2 z = \alpha z \neq 0$. This is impossible by a simple computation:

$$\phi_k^2 z = 2(x_k^2 + x_{k+1}^2) - (x_k^2 - x_{k+1}^2)^2 = 2(q(i_k) + q(i_{k+1})) - (q(i_k) - q(i_{k+1}))^2 = 0$$

on $M_{\underline{i}}$ since $i_k = i_{k+1} \pm 1$. This proves (2). \square

Corollary 7.9. *Assume that $\underline{i} = (i_1, \dots, i_n) \in \mathbb{Z}_+^n$ is a weight of an irreducible integral completely splittable $\mathcal{H}_n^{\text{aff}}$ -module M . If $i_k = i_{k+2}$ for some $1 \leq k \leq n-2$, then $i_k = i_{k+2} = 0$ and $i_{k+1} = 1$.*

Proof. If $i_k \neq i_{k+1} \pm 1$, then $s_k \cdot \underline{i}$ is a weight of M of the form (\dots, u, u, \dots) by Lemma 7.8(1), which contradicts Lemma 7.7. Hence $i_k = i_{k+1} \pm 1$. By Lemma 7.8(2), we have

$$\begin{aligned} (a-b)s_k z &= -((x_k + x_{k+1}) + c_k c_{k+1}(x_k - x_{k+1}))z, \\ (a-b)s_{k+1} z &= -((x_{k+1} + x_{k+2}) + c_{k+1} c_{k+2}(x_{k+1} - x_{k+2}))z, \end{aligned}$$

for $z \in M_{\underline{i}}$, where $a = q(i_k) = q(i_{k+2})$, $b = q(i_{k+1})$. A direct calculation shows that

$$\begin{aligned} & (a-b)(b-a)(a-b)(s_k s_{k+1} s_k - s_{k+1} s_k s_{k+1})z \\ &= ((x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2}))z \\ (7.18) \quad &= 0. \end{aligned}$$

for $z \in M_{\underline{i}}$ since $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$. Decompose $M_{\underline{i}}$ as $M_{\underline{i}} = N_1 \oplus N_2$, where $N_1 = \{z \in M_{\underline{i}} \mid x_k z = x_{k+2} z = \pm \sqrt{a} z\}$ and $N_2 = \{z \in M_{\underline{i}} \mid x_k z = -x_{k+2} z = \pm \sqrt{a} z\}$. Now applying the equality (7.18) to z in N_1 and N_2 , we obtain that

$$2\sqrt{q(i_k)}(6q(i_{k+1}) + 2q(i_k)) = 0,$$

which, thanks to $i_{k+1} = i_k \pm 1$, is equivalent to one of the following two identities:

$$(7.19) \quad \text{if } i_{k+1} = i_k - 1, \text{ then } \sqrt{i_k(i_k + 1)}(4i_k - 2)i_k = 0;$$

$$(7.20) \quad \text{if } i_{k+1} = i_k + 1, \text{ then } \sqrt{i_k(i_k + 1)}(4i_k + 6)(i_k + 1) = 0.$$

There is no solution for (7.19), and the solution of (7.20) is $i_k = 0, i_{k+1} = 1$. \square

Denote by $\mathcal{W}(n)$ the set of weights of all integral irreducible completely splittable $\mathcal{H}_n^{\text{aff}}$ -modules.

Proposition 7.10. *Assume $\underline{i} \in \mathcal{W}(n)$ and $i_k = i_\ell = a$ for some $1 \leq k < \ell \leq n$.*

- (1) *If $a = 0$, then $1 \in \{i_{k+1}, \dots, i_{\ell-1}\}$.*
- (2) *If $a \geq 1$, then $\{a-1, a+1\} \subseteq \{i_{k+1}, \dots, i_{\ell-1}\}$.*

Proof. Without loss of generality, we can assume that $a \notin \{i_{k+1}, \dots, i_{\ell-1}\}$.

If $a = 0$ but $1 \notin \{i_{k+1}, \dots, i_{\ell-1}\}$, we can repeatedly swap i_ℓ with $i_{\ell-1}$ then with $i_{\ell-2}$, etc., all the way to obtain a weight of M of the form $(\dots, 0, 0, \dots)$ by Lemma 7.8. This contradicts Lemma 7.7. This proves (1).

Now assume $a \geq 1$ and $a+1 \notin \{i_{k+1}, \dots, i_{\ell-1}\}$. If $a-1$ does not appear between i_{k+1} and $i_{\ell-1}$ in \underline{i} , then we can swap i_ℓ with $i_{\ell-1}$ then with $i_{\ell-2}$, etc., and by Lemma 7.8 this gives rise to a weight of M having the form (\dots, a, a, \dots) , which contradicts Lemma 7.7. If $a-1$ appears only once between i_{k+1} and $i_{\ell-1}$ in \underline{i} , then again by swapping i_ℓ with $i_{\ell-1}$ then with $i_{\ell-2}$, etc. we obtain a weight of M of the form $(\dots, a, a-1, a, \dots)$, which contradicts Corollary 7.9. Hence $a-1$ appears at least twice between i_{k+1} and $i_{\ell-1}$ in \underline{i} . This implies that there exist $k < k_1 < \ell_1 < \ell$ such that

$$i_{k_1} = i_{\ell_1} = a-1, \{a, a-1\} \cap \{i_{k_1+1}, \dots, i_{\ell_1-1}\} = \emptyset.$$

An identical argument shows that there exist $k_1 < k_2 < \ell_2 < \ell_1$ such that

$$i_{k_2} = i_{\ell_2} = a-2, \{a, a-1, a-2\} \cap \{i_{k_2+1}, \dots, i_{\ell_2-1}\} = \emptyset.$$

Continuing in this way, we obtain $k < s < t < l$ such that

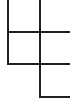
$$i_s = i_t = 0, \{a, a-1, \dots, 1, 0\} \cap \{i_{s+1}, \dots, i_{t-1}\} = \emptyset,$$

which contradicts (1).

Now assume that $a \geq 1$ and $a-1 \notin \{i_{k+1}, \dots, i_{l-1}\}$. Then $a+1$ must appear in the subsequence $(i_{k+1}, \dots, i_{l-1})$ at least twice, otherwise we can repeatedly swap i_l with i_{l-1} then with i_{l-2} , etc., all the way to obtain a weight of M of the form $(\dots, a, a+1, a \dots)$ by Lemma 7.8, which contradicts Corollary 7.9. Continuing this way we see that any integer greater than a will appear in the finite sequence $(i_{k+1}, \dots, i_{l-1})$ which is impossible. This completes the proof of (2). \square

For $\nu, \xi \in \mathcal{SP}$ such that $\nu \subseteq \xi$, the diagram obtained by removing the subdiagram ν^* from the shifted diagram ξ^* is called a *skew shifted diagram* and denoted by ξ/ν . It is possible that a skew shifted diagram is realized by two different pairs $\nu \subseteq \xi$ and $\tilde{\nu} \subseteq \tilde{\xi}$.

Example 7.11. Assume $\xi = (5, 3, 2, 1)$ and $\nu = (5, 1)$. Then the corresponding skew shifted Young diagram ξ/ν is



A filling by $1, 2, \dots, n$ in a skew shifted diagram ξ/ν with $|\xi/\nu| = n$ such that the entries strictly increase from left to right along each row and down each column is called a *standard skew shifted tableau* of size n . Denote

$$\begin{aligned} \mathcal{W}'(n) &= \{\underline{i} \in \mathbb{Z}_+^n \text{ satisfying the properties in Proposition 7.10}\}, \\ \mathcal{F}(n) &= \{\text{standard skew shifted tableaux of size } n\}. \end{aligned}$$

Proposition 7.12. *There exists a canonical bijection between $\mathcal{W}'(n)$ and $\mathcal{F}(n)$.*

Proof. For $T \in \mathcal{F}(n)$, set

$$c(T) = (c(T_1), c(T_2), \dots, c(T_n)) \in \mathbb{Z}_+^n,$$

where $c(T_k)$ denotes the content of the cell occupied by k in T , for $1 \leq k \leq n$. It is easy to show that $c(T) \in \mathcal{W}'(n)$. Then we define

$$(7.21) \quad \Theta : \mathcal{F}(n) \longrightarrow \mathcal{W}'(n), \quad \Theta(T) = c(T).$$

To show that Θ is a bijection, we shall construct by induction on n a unique tableau $T(\underline{i}) \in \mathcal{F}(n)$ satisfying $\Theta(T(\underline{i})) = \underline{i}$, for a given $\underline{i} = (i_1, \dots, i_n) \in \mathcal{W}'(n)$. If $n = 1$, let $T(\underline{i}) \in \mathcal{F}(n)$ be a cell labeled by 1 of content i_1 . Assume that $T(\underline{i}') \in \mathcal{F}(n-1)$ is already defined, where $\underline{i}' = (i_1, \dots, i_{n-1}) \in \mathcal{W}'(n-1)$. Set $u = i_n$.

Case 1: $T(\underline{i}')$ contains neither a cell of content $u-1$ nor a cell of content $u+1$. Adding a new component consisting of one cell labeled by n of content u to T' , we obtain a new standard tableau $T \in \mathcal{F}(n)$. Set $T(\underline{i}) = T$.

Case 2: $T(\underline{i}')$ contains cells of content $u-1$ but no cell of content $u+1$. This implies $u+1 \notin \{i_1, \dots, i_n\}$. Since (i_1, \dots, i_n) belongs to $\mathcal{W}'(n)$, u does not appear in \underline{i}' and hence $u-1$ appears only once in \underline{i}' by Proposition 7.10. Therefore there is no cell of content u and only one cell denoted by A of content $u-1$ in $T(\underline{i}')$. So we can add a new cell labeled by n with content u to the right of A to obtain a new tableau T . Set

$T(\underline{i}) = T$. Observe that there is no cell above A in the column containing A since there is no cell of content u in $T(\underline{i}')$. Hence $T(\underline{i}) \in \mathcal{F}(n)$.

Case 3: $T(\underline{i}')$ contains cells of content $u + 1$ but no cell of content $u - 1$. This implies $u - 1 \notin \{i_1, \dots, i_n\}$. Since (i_1, \dots, i_n) is in $\mathcal{W}'(n)$, u does not appear in \underline{i}' and hence $u + 1$ appears only once in \underline{i}' by Proposition 7.10. Therefore $T(\underline{i}')$ contains only one cell denoted by B of content $u + 1$ and no cell of content u . This means that there is no cell below B in $T(\underline{i}')$. Adding a new cell labeled by n of content u below B , we obtain a new tableau T . Set $T(\underline{i}) = T$. Clearly $T(\underline{i}) \in \mathcal{F}(n)$.

Case 4: $T(\underline{i}')$ contains cells of contents $u - 1$ and $u + 1$. Let C and D be the last cells on the diagonals of content $u - 1$ and $u + 1$, respectively. Suppose that C is labeled by s and D is labeled by t . Then $i_s = u - 1, i_t = u + 1$, and moreover $u - 1 \notin \{i_{t+1}, \dots, i_{n-1}\}, u + 1 \notin \{i_{s+1}, \dots, i_{n-1}\}$. Since $i_n = u$, by Proposition 7.10 we see that $u \notin \{i_{t+1}, \dots, i_{n-1}\}$ and $u \notin \{i_{s+1}, \dots, i_{n-1}\}$. This implies that there is no cell below C and no cell to the right of D in $T(\underline{i}')$. Moreover C and D must be of the following shape

$$\begin{array}{|c|} \hline C \\ \hline D \\ \hline \end{array}.$$

Add a new cell labeled by n to the right of D and below C to obtain a new tableau T . Set $T(\underline{i}) = T$. Again it is clear that $T(\underline{i}) \in \mathcal{F}(n)$. \square

Example 7.13. Suppose $n = 5$. Then the standard skew shifted tableau corresponding to $\underline{i} = (1, 2, 0, 1, 0) \in \mathcal{W}'(5)$ is

$$T(\underline{i}) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline & 5 \\ \hline \end{array}.$$

7.5. Classification of irreducible completely splittable $\mathcal{H}_n^{\text{aff}}$ -modules. For a skew shifted diagram ξ/ν of size n , denote by $\mathcal{F}(\xi/\nu)$ the set of standard skew shifted tableaux of shape ξ/ν , and form a vector space

$$\widehat{U}^{\xi/\nu} = \bigoplus_{T \in \mathcal{F}(\xi/\nu)} \mathcal{C}l_n v_T.$$

Define

$$(7.22) \quad x_i v_T = \sqrt{q(c(T_i))} v_T, \quad 1 \leq i \leq n,$$

$$(7.23) \quad \begin{aligned} s_k v_T &= \left(\frac{1}{\sqrt{q(c(T_{k+1}))} - \sqrt{q(c(T_k))}} + \frac{1}{\sqrt{q(c(T_{k+1}))} + \sqrt{q(c(T_k))}} c_k c_{k+1} \right) v_T \\ &+ \sqrt{1 - \frac{2(q(c(T_{k+1})) + q(c(T_k)))}{(q(c(T_{k+1})) - q(c(T_k)))^2}} v_{s_k T}, \quad 1 \leq k \leq n-1, \end{aligned}$$

where $s_k T$ denotes the tableau obtained by switching k and $k + 1$ in T and $v_{s_k T} = 0$ if $s_k T$ is not standard.

Proposition 7.14. Suppose ξ/ν is a skew shifted diagram of size n . Then $\widehat{U}^{\xi/\nu}$ affords a completely splittable $\mathcal{H}_n^{\text{aff}}$ -module under the action defined by (7.22) and (7.23).

Proof. We check the defining relations (2.1), (2.4), (7.5), and (7.6). It is routine to check (7.5), (7.6) and (2.4). It remains to check the Coxeter relations (2.1).

It is clear by (7.6) that $s_k s_l = s_l s_k$ if $|l - k| > 1$. We now prove $s_k^2 = 1$. Let $T \in \mathcal{F}(\xi/\nu)$. A direct calculation shows that if $s_k T$ is standard then

$$s_k^2 v_T = \left(\frac{2(q(c(T_{k+1})) + q(c(T_k)))}{(q(c(T_{k+1})) - q(c(T_k)))^2} \right) v_T + \left(1 - \frac{2(q(c(T_{k+1})) + q(c(T_k)))}{(q(c(T_{k+1})) - q(c(T_k)))^2} \right) v_T = v_T.$$

Otherwise, if $s_k T$ is not standard then $c(T_k) = c(T_{k+1}) \pm 1$, and we have

$$s_k^2 v_T = \left(\frac{2(q(c(T_{k+1})) + q(c(T_k)))}{(q(c(T_{k+1})) - q(c(T_k)))^2} \right) v_T = v_T.$$

So it remains to prove that $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$. Fix $1 \leq k \leq n - 2$ and $T \in \mathcal{F}(\xi/\nu)$. Let $a = q(c(T_k))$, $b = q(c(T_{k+1}))$, $c = q(c(T_{k+2}))$. If $c(T_k) = c(T_{k+2})$, then by Corollary 7.9 we have $c(T_k) = c(T_{k+2}) = 0$, $c(T_{k+1}) = 1$ and hence $a = c = 0$, $b = 2$. Then $(a - b)^2 = 2(a + b)$. By (7.23), we obtain that

$$s_k v_T = \frac{\sqrt{2}}{2} (1 + c_k c_{k+1}) v_T, \quad s_{k+1} v_T = \frac{\sqrt{2}}{2} (-1 + c_{k+1} c_{k+2}) v_T.$$

Then one can check that $s_k s_{k+1} s_k v_T = s_{k+1} s_k s_{k+1} v_T$.

Now assume $c(T_k) \neq c(T_{k+2})$ and hence a, b, c are distinct. Then it suffices to show $\phi_k \phi_{k+1} \phi_k v_T = \phi_{k+1} \phi_k \phi_{k+1} v_T$ for the intertwining elements ϕ_k, ϕ_{k+1} defined via (7.9). It is clear by (7.23) that

$$\phi_r v_T = \sqrt{(q(c(T_{r+1})) - q(c(T_r)))^2 - 2(q(c(T_{r+1})) + q(c(T_r)))} v_{s_r T},$$

if $s_r T$ is standard and $\phi_r v_T = 0$ otherwise for $1 \leq r \leq n - 1$. Now for our fixed $1 \leq k \leq n - 2$, if one of $c(T_k) - c(T_{k+1})$, $c(T_{k+1}) - c(T_{k+2})$ and $c(T_k) - c(T_{k+2})$ is ± 1 , then $\phi_k \phi_{k+1} \phi_k v_T = 0 = \phi_{k+1} \phi_k \phi_{k+1} v_T$. Otherwise, one can check that

$$\begin{aligned} \phi_k \phi_{k+1} \phi_k v_T &= \left(\sqrt{(a - b)^2 - 2(a + b)} \sqrt{(b - c)^2 - 2(b + c)} \sqrt{(a - c)^2 - 2(a + c)} \right) v_T \\ &= \phi_{k+1} \phi_k \phi_{k+1} v_T. \end{aligned}$$

Therefore the proposition is proved. \square

For a skew shifted diagram ξ/ν of size n , pick a standard skew shifted tableau $T^{\xi/\nu}$ of shape ξ/ν . Observe that the \mathcal{P}_n^c -module $\mathcal{C}l_n v_{T^{\xi/\nu}}$ contains an irreducible submodule $\mathcal{L}(\xi/\nu)$ which is isomorphic to $L(c(T_1^{\xi/\nu})) \otimes L(c(T_2^{\xi/\nu})) \otimes \cdots \otimes L(c(T_n^{\xi/\nu}))$ and moreover

$$(7.24) \quad \mathcal{C}l_n v_{T^{\xi/\nu}} \cong (\mathcal{L}(\xi/\nu))^{\oplus 2^{\lfloor \frac{\ell(\xi) - \ell(\nu)}{2} \rfloor}}.$$

Set

$$U^{\xi/\nu} := \sum_{\sigma \in \mathfrak{S}_n} \phi_\sigma \mathcal{L}(\xi/\nu) \subseteq \widehat{U}^{\xi/\nu},$$

where $\phi_\sigma = \phi_{i_1} \phi_{i_2} \cdots \phi_{i_k}$ with a reduced expression $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$.

Lemma 7.15. *Suppose ξ/ν is a skew shifted diagram of size n . Then $U^{\xi/\nu}$ is a $\mathcal{H}_n^{\text{aff}}$ -submodule of $\widehat{U}^{\xi/\nu}$.*

Proof. Clearly, $U^{\xi/\nu}$ is a \mathcal{P}_n^c -submodule of $\widehat{U}^{\xi/\nu}$ by (7.11) and (7.12). Let $\sigma \in \mathfrak{S}_n$ and $z \in \mathcal{L}(\xi/\nu)$ be such that $\phi_\sigma z \neq 0$. Then

$$\phi_k \phi_\sigma z = (s_k(x_k^2 - x_{k+1}^2) + (x_k + x_{k+1}) + c_k c_{k+1}(x_k - x_{k+1})) \phi_\sigma z \in U^{\xi/\nu}.$$

Meanwhile $(x_k^2 - x_{k+1}^2)$ acts as a nonzero scalar on $\phi_\sigma z$ and hence $s_k \phi_\sigma z \in U^{\xi/\nu}$. \square

The following theorem is due independently to [HKS, Wan]. The results of the paper of the first author [Wan] were actually formulated and established over any characteristic $p \neq 2$.

Theorem 7.16. *Suppose ξ/ν and ξ'/ν' are skew shifted diagrams of size n . Then*

- (1) $U^{\xi/\nu}$ is an irreducible $\mathcal{H}_n^{\text{aff}}$ -module.
- (2) $U^{\xi/\nu} \cong U^{\xi'/\nu'}$ if and only if $\xi/\nu = \xi'/\nu'$.
- (3) $\widehat{U}^{\xi/\nu} \cong (U^{\xi/\nu})^{\oplus 2^{\lfloor \frac{\ell(\xi) - \ell(\nu)}{2} \rfloor}}$.
- (4) $\dim U^{\xi/\nu} = 2^{n - \lfloor \frac{\ell(\xi) - \ell(\nu)}{2} \rfloor} g^{\xi/\nu}$, where $g^{\xi/\nu}$ denotes the number of standard skew shifted tableaux of shape ξ/ν .
- (5) Every integral irreducible completely splittable $\mathcal{H}_n^{\text{aff}}$ -module is isomorphic to $U^{\xi/\nu}$ for some skew shifted diagram ξ/ν of size n .

Proof. Suppose N is a nonzero submodule of $U^{\xi/\nu}$. Then $N_{\underline{i}} \neq 0$ for some $\underline{i} = \sigma \cdot c(T^{\xi/\nu})$ and $\sigma \in \mathfrak{S}_n$, and hence $N_{c(T^{\xi/\nu})} \neq 0$. Observe that $U_{c(T^{\xi/\nu})}^{\xi/\nu} \cong \mathcal{L}(\xi/\nu)$. This implies that $N_{c(T^{\xi/\nu})} = U_{c(T^{\xi/\nu})}^{\xi/\nu}$ as $\mathcal{L}(\xi/\nu)$ is irreducible as \mathcal{P}_n^c -module. Therefore $N = U^{\xi/\nu}$. This proves (1). If $U^{\xi/\nu} \cong U^{\xi'/\nu'}$, then $T^{\xi/\nu} \in \mathcal{F}(\xi'/\nu')$. Hence, $\xi/\nu = \xi'/\nu'$ and whence (2). Part (3) follows by the definition of $\widehat{U}^{\xi/\nu}$ and (7.24), and (4) follows from (3).

It remains to prove (5). Suppose U is an integral irreducible completely splittable $\mathcal{H}_n^{\text{aff}}$ -module and let $u_{\underline{i}}$ be a non-zero weight vector of U . By Propositions 7.10 and 7.12, there exists $T \in \mathcal{F}(n)$ such that $\underline{i} = c(T)$. Assume T is of shape ξ/ν . Observe that there always exists a sequence of simple transpositions s_{k_1}, \dots, s_{k_r} such that $s_{k_j} \cdots s_{k_1} T$ is standard for $1 \leq j \leq r$ and $s_{k_r} \cdots s_{k_1} T = T^{\xi/\nu}$. Then it follows by Lemma 7.8 that $u_{\xi/\nu} := \phi_{s_{k_r}} \cdots \phi_{s_{k_1}} u_{\underline{i}}$ is a non-zero weight vector of U of weight $c(T^{\xi/\nu})$. Hence $U_{c(T^{\xi/\nu})} \neq 0$ and it must contain a \mathcal{P}_n^c -submodule U' isomorphic to $\mathcal{L}(\xi/\nu)$. Again by Lemma 7.8, $\sum_{\sigma \in \mathfrak{S}_n} \phi_\sigma U'$ forms a $\mathcal{H}_n^{\text{aff}}$ -submodule of U . Thus $U = \sum_{\sigma \in \mathfrak{S}_n} \phi_\sigma U'$. Let $\tau : U' \rightarrow \mathcal{L}(\xi/\nu)$ be a \mathcal{P}_n^c -module isomorphism. Then it is easy to check that the map $\bar{\tau} : \sum_{\sigma \in \mathfrak{S}_n} \phi_\sigma U' \rightarrow U^{\xi/\nu}$, which sends $\phi_\sigma z$ to $\phi_\sigma \tau(z)$ for all $z \in U'$, is an $\mathcal{H}_n^{\text{aff}}$ -module isomorphism. \square

7.6. The seminormal form construction for \mathcal{H}_n . When restricting Theorem 7.16 to the case of shifted diagrams, we have the following.

Theorem 7.17. $\{U^\xi | \xi \in \mathcal{SP}_n\}$ forms a complete set of non-isomorphic irreducible \mathcal{H}_n -modules. The Jucys-Murphy elements J_1, J_2, \dots, J_n act semisimply on each U^ξ .

Proof. Consider the $\mathcal{H}_n^{\text{aff}}$ -modules \widehat{U}^ξ and U^ξ , for $\xi \in \mathcal{SP}_n$. For any standard shifted tableau T of shape ξ , we have $c(T_1) = 0$ and hence $x_1 v_T = 0$. Hence the action of

$\mathcal{H}_n^{\text{aff}}$ on \widehat{U}^ξ and U^ξ factors through to an action of \mathcal{H}_n and x_k acts as J_k by (7.8), as $\mathcal{H}_n \cong \mathcal{H}_n^{\text{aff}}/\langle x_1 \rangle$. The theorem now follows from Theorem 7.16. \square

The construction of \mathcal{H}_n -modules U^ξ above can be regarded a seminormal form for irreducible \mathcal{H}_n -modules. Theorem 7.17 in different forms has been established via different approaches in [Naz, VS, HKS, Wan].

REFERENCES

- [BL] F. Bergeron, and F. Lamontagne, *Decomposition of the diagonal action of S_n on the coinvariant space of $S_n \times S_n$* , Sémin. Lothar. Combin. **52** (2004/07), Art. B52e, 24 pp.
- [Br] R. Brylinski, *Limits of weight spaces, Lusztig's q -analogs, and fiberings of adjoint orbits*, J. Amer. Math. Soc. **2** (1989), 517–533.
- [BK] J. Brundan and A. Kleshchev, *Hecke-Clifford superalgebras, crystals of type $A_{2l}^{(2)}$, and modular branching rules for \widehat{S}_n* , Represent. Theory **5** (2001), 317–403.
- [Ch] I. Cherednik, *A new interpretation of Gel'fand-Tsetlin bases*, Duke Math. J. **54** (1987), 563–577.
- [CW] S.-J. Cheng and W. Wang, *Dualities and representations of Lie superalgebras*, book in preparation, Chapter 3 on Schur Duality and Appendix A on Symmetric Functions, available at the webpage <http://www.math.virginia.edu/~ww9c/>
- [DLT] J. Désarménien, B. Leclerc and J. Thibon, *Hall-Littlewood functions and Kostka-Foulkes polynomials in representation theory*, Sémin. Lothar. Combin. **32** (1994), Art. B32c, 38 pp.
- [GP] A. Garsia and C. Procesi, *On certain graded S_n -Modules and the q -Kostka polynomials*, Adv. Math. **94** (1992), 82–138.
- [GJK] D. Grantcharov, J.-H. Jung, S.-J. Kang, M. Kashiwara and M. Kim, *Crystal bases for the quantum queer superalgebra and semistandard decomposition tableaux*, arXiv:1103.1456, 2011.
- [HKS] D. Hill, J. Kujawa and J. Sussan, *Degenerate affine Hecke-Clifford algebras and type Q Lie superalgebras*, Math. Z **268** (2011), 1091–1158.
- [JN] A. Jones and M. Nazarov, *Affine Sergeev algebra and q -analogues of the Young symmetrizers for projective representations of the symmetric group*, Proc. London Math. Soc. **78** (1999), 481–512.
- [Jo] T. Józefiak, *Semisimple superalgebras*, In: Algebra—Some Current Trends (Varna, 1986), pp. 96–113, Lect. Notes in Math. **1352**, Springer-Berlag, Berlin-New York, 1988.
- [Jo1] T. Józefiak, *Characters of projective representations of symmetric groups*, Expo. Math. **7** (1989), 193–247.
- [Jo2] T. Józefiak, *A class of projective representations of hyperoctahedral groups and Schur Q -functions*, Topics in Algebra, Banach Center Publ., **26**, Part 2, PWN-Polish Scientific Publishers, Warsaw (1990), 317–326.
- [Ka] R. Kane, *Reflection groups and invariant theory*, Springer-Verlag, New York, 2001.
- [Ki] A.A. Kirillov, *Polynomial covariants of the symmetric group and some of its analogues*, Funct. Anal. Appl. **18** (1984), 63–64.
- [KP] A.A. Kirillov and I. Pak, *Covariants of the symmetric group and its analogues in Weyl algebras*, Funct. Anal. Appl. **24** (1990), 172–176.
- [KW] T. Khongsap and W. Wang, *Hecke-Clifford algebras and spin Hecke algebras I: the classical affine type*, Transformation Groups **13** (2008), 389–412.
- [Kle] A. Kleshchev, *Linear and Projective Representations of Symmetric Groups*, Cambridge University Press, 2005.
- [LS] A. Lascoux and M. Schützenberger, *Sur une conjecture de H.O. Foulkes*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), A323–A324.
- [Lu1] G. Lusztig, *Irreducible representations of finite classical groups*, Invent. Math. **43** (1977), 125–175.

- [Lu2] G. Lusztig, *Singularities, character formulas, and a q -analogue of weight multiplicities*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque **101–102** (1983), 208–229.
- [Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Clarendon Press, Oxford, 1995.
- [Naz] M. Nazarov, *Young's symmetrizers for projective representations of the symmetric group*, Adv. Math. **127** (1997), 190–257.
- [OV] A. Okounkov and A. Vershik, *A new approach to representation theory of symmetric groups*, Selecta Math. (N.S) **2** (1996), 581–605.
- [Ol] G. Olshanski, *Quantized universal enveloping superalgebra of type Q and a super-extension of the Hecke algebra*, Lett. Math. Phys. **24** (1992), 93–102.
- [Ram] A. Ram, *Skew shape representations are irreducible*, Combinatorial and geometric representation theory (Seoul, 2001), Contemp. Math., **325**, Amer. Math. Soc., Providence, RI, (2003), 161–189.
- [Re] E. Read, *The α -regular classes of the generalized symmetric groups*, Glasgow Math. J. **17** (1976), 144–150.
- [Ro] H. Rosengren, *Schur Q -polynomials, multiple hypergeometric series and enumeration of marked shifted tableaux*, J. Combin. Theory Ser. A **115** (2008), 376–406.
- [Ru] O. Ruff, *Completely splittable representations of symmetric groups and affine Hecke algebras*, J. Algebra **305** (2006), 1197–1211.
- [Sag] B. Sagan, *Shifted tableaux, Schur Q -functions, and a conjecture of R. Stanley*, J. Combin. Theory Ser. A **45** (1987), 62–103.
- [Sch] I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **139** (1911), 155–250.
- [Se1] A. Sergeev, *Tensor algebra of the identity representation as a module over the Lie superalgebras $GL(n, m)$ and $Q(n)$* , Math. USSR Sbornik **51** (1985), 419–427.
- [Se2] A. Sergeev, *The Howe duality and the projective representation of symmetric groups*, Repr. Theory **3** (1999), 416–434.
- [Ser] L. Serrano, *The shifted plactic monoid*, Math. Z. **266** (2010), 363–392.
- [S] R. Steinberg, *A geometric approach to the representations of the full linear group over a Galois field*, Trans. Amer. Math. Soc. **71** (1951), 274–282.
- [St] J. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), 87–134.
- [TZ] G. Tudose and M. Zabrocki, *A q -analogue of Schur's Q -functions*, In: Algebraic combinatorics and quantum groups, 135–161, World Sci. Publ., River Edge, NJ, 2003.
- [VS] A. Vershik and A. Sergeev, *A new approach to the representation theory of the symmetric groups. IV. Z_2 -graded groups and algebras: projective representations of the group S_n* , Mosc. Math. J. **8** (2008), no. 4, 813–842.
- [Wan] J. Wan, *Completely splittable representations of affine Hecke-Clifford algebras*, J. Algebraic Combin. **32** (2010), 15–58.
- [WW1] J. Wan and W. Wang, *Spin invariant theory for the symmetric group*, J. Pure Appl. Algebra **215** (2011), 1569–1581.
- [WW2] J. Wan and W. Wang, *Spin Kostka polynomials*, arXiv:1102.4901, 2011.
- [Wa1] W. Wang, *Double affine Hecke algebras for the spin symmetric group*, Math. Res. Lett. **16** (2009), 1071–1085.
- [Wa2] W. Wang, *Spin Hecke algebras of finite and affine types*, Adv. in Math. **212** (2007), 723–748.
- [Wor] D. Worley, *A theory of shifted Young tableaux*, Ph.D. thesis, MIT, 1984.
- [Ya] M. Yamaguchi, *A duality of the twisted group algebra of the symmetric group and a Lie superalgebra*, J. Alg. **222** (1999), 301–327.
- [You] Y. You, *Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups*, Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), 449–464, Adv. Ser. Math. Phys., **7** (1989), World Sci. Publ., Teaneck, NJ.
- [Ze] A. Zelevinsky, *Representations of finite classical groups. A Hopf algebra approach*, Lect. Notes in Math. **869**, Springer-Verlag, Berlin-New York, 1981.

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